Projective Geometric Computing

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Abstract

1 Applications to Projective Geometry

This paper applies geometric algebra to the geometry of conics in the plane. Starting from the classical double algebra expression for a conic on 5 points $P_i \in P^2(R)$ in terms of a running variable, we show how to eliminate this variable (by the use of tensor products) and express the conic on 5 points without resorting to a running variable. Writing $P = \{P_1, P_2, P_3, P_4, P_5\}$, and designating the conic by Q_P, the homogeneous point equation of Q_P can be expressed as $Q_P(X) = 0$, where

$$\begin{aligned} Q_P(X) = {} & (P_1 \wedge P_3 \wedge P_5)(P_2 \wedge P_4 \wedge P_5)(P_2 \wedge P_3 \wedge X)(P_1 \wedge P_4 \wedge X) - \\ & (P_2 \wedge P_3 \wedge P_5)(P_1 \wedge P_4 \wedge P_5)(P_1 \wedge P_3 \wedge X)(P_2 \wedge P_4 \wedge X) \end{aligned} \tag{1}$$

Making use of tensor products, we show that $Q_P(X) \equiv B_P(X, X)$ where

$$\begin{aligned} B_P = {} & (P_1 \wedge P_3 \wedge P_5)(P_2 \wedge P_4 \wedge P_5)(P_2 \wedge P_3)_\Lambda \otimes (P_1 \wedge P_4)_\Lambda - \\ & (P_2 \wedge P_3 \wedge P_5)(P_1 \wedge P_4 \wedge P_5)(P_1 \wedge P_3)_\Lambda \otimes (P_2 \wedge P_4)_\Lambda \end{aligned} \tag{2}$$

Here $(P_2 \wedge P_3)_\Lambda \otimes (P_1 \wedge P_4)_\Lambda$ denotes the function on $\mathbf{P}^2$x$\mathbf{P}^2$ taking (X, Y) to

$$((P_2 \wedge P_3)_\Lambda \otimes (P_1 \wedge P_4)_\Lambda)(X, Y) = (P_2 \wedge P_3 \wedge X)(P_1 \wedge P_4 \wedge Y) \tag{3}$$

with the analogous expression for the tensor product in the second term of (2). Differentiating this expression with respect to one of the participating points leads to a formula that expresses the sensitivity of a conic with respect to small changes in one of its points.

The expression (2) for a conic as a function of five points, opens up possibilities for many interesting types of computations involving conics. We make use of it and apply the unified geo-MAP computational technique[1] in order to deduce the following

Prop. 1: The conic that passes through two given points p and q and osculates a given curve M at a point $m(s)$ is given by the equation

$$((x - q) \wedge m')(p \wedge x) = ((x - p) \wedge m')(q \wedge x) \tag{4}$$

where $m' = m'(s)$.

1. Naeve&Svensson: *Geo-MAP Unification*, to be published in *Geometric Computing with Clifford Algebra*, (ed. G. Sommer), Springer, 2000.

2 Polarizing with respect to a conic

Let $T : V \to V$ be a symmetric linear mapping, and consider the conic $Q = \{x \in V : x * Tx = 0\}$. The *polar* of the point x with respect to T (or Q) is defined by

$$Pol(x;T) = T(x)I^{-1} = \overline{T(x)} \tag{5}$$

Notation: The large over bar should really be a 'tilde', but our word-processing program does not support 'large tildes'

Letting $\underline{T}$ denote the outermorphism induced by T, we deduce the following generalized form of the pole/polar relationship of a point and a line with respect to a conic.

Prop. 2: For each symmetric linear $T : G_1 \to G_1$ and each $x \in G$ we have

$$(\det \underline{T}) x * \underline{T}(x) = 0 \quad \Leftrightarrow \quad \overline{\underline{T}(x)} * \underline{T}(\overline{\underline{T}(x)}) = 0 \tag{6}$$

Note: The relationship (6) is valid also for rank deficient conics.

Let I be a unit n-blade, and let $\tilde{x} = xI^{\dagger}$. The dual-outer product can then be expressed - for general multivectors x and y - in the following way:

$$x \vee y = (\tilde{x} \wedge \tilde{y})I \tag{7}$$

We call a mapping $F : G \to G$ a *dual-outermorphism* if
(i): F is linear. (ii): F is step-preserving. (iii): $F(I) = I$.
(vi): $F(x \vee y) = F(x) \vee F(y)$ for general multivectors x and y.

3 Reciprocating a conic with respect to another conic

In §(3.3) we show how to perform *reciprocation* (dualization) of a conic with respect to another conic by polarizing the former with respect to the latter. This mapping has a natural expression in terms of outer- and dual-outermorphisms, as given by

Prop. 3: Let S and T be symmetric linear mappings $G^1 \to G^1$ with T non-singular. Polarizing the points of the conic $x * S(x) = 0$ with respect to the conic $x * T(x) = 0$ gives rise to the conic

$$y * (\underline{T} \circ S_{\vee} \circ \underline{T}(y)) = 0 \tag{8}$$

where $y = \overline{T(x)}$ is the polar of x with respect to T.

4 Generalized complex numbers and geometric duality

We demonstrate an interesting type of isomorphism between two vector spaces, using two different kinds of generalized complex numbers. This isomorphism has potential interest as a means of representing the interplay between real and complex projective geometry in 3 dimensions.

Consider the geometric algebra $G = \mathbf{R}(e_1, e_2, e_3)$ with unit pseudo-scalar $I = e_1 e_2 e_3$, and let $\mathbf{C}$ denote the ordinary complex numbers with imaginary unit i. Then I commutes with each element of G, $I^2 = -1$, $I^{-1} = -I$, and we have

$$\tilde{1} = 1I^{-1} = -I \quad , \qquad \tilde{1}e_k = \tilde{e}_k \quad , \qquad \tilde{1}\tilde{e}_k = -e_k \tag{9}$$

We define the following two versions of generalized complex numbers:

$$C_0 = G_0 + G_3 \quad , \qquad C_1 = G_1 + G_2 \tag{10}$$

It is clear that the following mapping mapping is an algebra isomorphism:

$$\begin{aligned} C_0 &\rightarrow \mathbf{C} \\ x + y\tilde{1} &\rightarrow x + iy \end{aligned} \tag{11}$$

Noting that C_1 is a vector space over $\mathbf{C}$, we consider the mapping $C_1 \rightarrow \mathbf{C}^3$ defined by

$$C_1 \ni \sum_{k=1}^{3} (x_k e_k + y_k \tilde{e}_k) \rightarrow \begin{bmatrix} x_1 + iy_1 \\ x_2 + iy_2 \\ x_3 + iy_3 \end{bmatrix} \in \mathbf{C}^3 \tag{12}$$

and show that this mapping is a vector space isomorphism, i.e. that

Prop. 4: C_1 and $\mathbf{C}^3$ are isomorphic as vector spaces over C_0 (or $\mathbf{C}$).

We feel that this isomorphism will be important in exploring the connection between real and complex 3-dimensional geometry, in a way that generalizes Felix Klein's classical embedding of $\mathbf{R}^2$ into $\mathbf{C}^2$. This representation can be used in order to keep track of complex conjugate aspects of real geometric configurations. It has been used for this purpose by Winroth in the dynamic geometry program *pdb*, developed as part of his thesis work[1] at the Computational Vision and Active Perception Laboratory (CVAP) at The Royal Institute of Technology (KTH). This program will be used in the presentation to visualize the mappings of polarization and reciprocation.

1. Winroth, H., *Dynamic Projective Geometry*, TRITA-NA-99/01, Dissertation, March 1999, Computational Vision and Active Perception Laboratory, NADA / KTH