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PROJECTIVE LINE GEOMETRY OF THE VISUAL OPERATOR

by

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TRITA-NA-8606

CVAP 29

Report from Computer Vision and Associative Pattern Processing Laboratory

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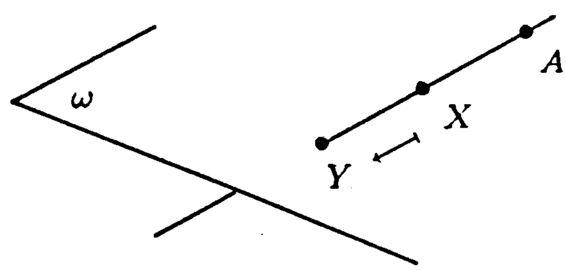
Introduction

Mathematical modelling of the visual process was the starting point of projective geometry 500 years ago. In this paper we return to this problem and, following Plücker [13] and Grassman [5] we develop a line geometric representation of the visual operator S regarded as the projection of $\mathbb{P}^3(\mathbb{R})\setminus\{A\}$ from the point A (the pinhole lense) onto the plane ω (the retina) induced by the set of lines through A.

This is the natural first order approximation of monocular vision and it is usually referred to as *the perspective transformation* (to be carefully distinguished from a perspectivity in projective geometry).



Since edge detection is of fundamental importance for the visual process and since linear edges abund in a world of man-made objects, the line geometric representation of S provides a natural framework for studying many of the visual inverse problems that arise in the recovery of 3D information from 2D images.

For the sake of convenience we state in this paper the necessary background material from projective geometry. Most of it is presented in any standard textbook, e.g. [2], [4] or [20]. The line geometric prerequisites are partly developed here and partly stated as facts. The reader is referred to [8], [9] or [15] for further information on this subject.

Real projective space

The real projective space $\mathbb{P}^3(\mathbb{R})$ or $\mathbb{P}^3$ for short is constructed from the ordinary affine 3-space $\mathbb{E}^3$ by the adjunction of a new element π_∞ called *the plane at infinity*, which intersects each of the old planes in its corresponding *line at infinity*, and each of the old lines in its corresponding *point at infinity*.

The affine plane π with its added line at infinity is called *the projective plane* π, and the affine line $\mathbf{p}$ with its added point at infinity is called *the projective line* $\mathbf{p}$.

Since nearly all lines and planes considered here will be projective, we will often drop this adjective and refer to them simply as *the line* $\mathbf{p}$ and *the plane* π when they are considered as embedded in $\mathbb{P}^3$ and as $\mathbb{P}^1$ and $\mathbb{P}^2$ when they are considered intrinsically.

The projective plane $\mathbb{P}^2$

The need to projectify the affine plane arose from studying the painter's problem: when an open planelike landscape was depicted on a canvas, certain points appeared out of nowhere, forming a line of *vanishing* points – the *horizon* line.

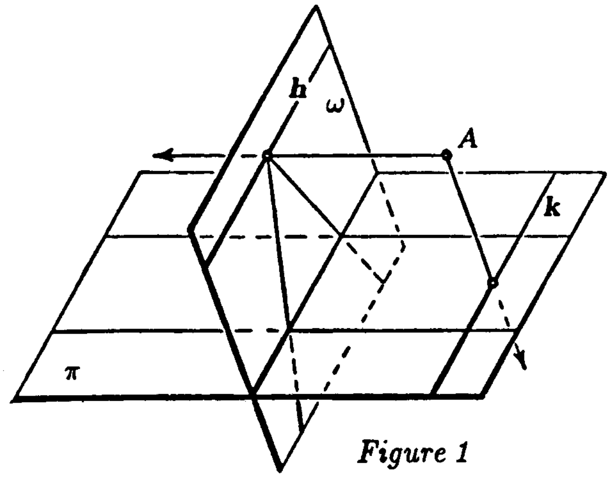


Figure 1

The mathematical model of this process – mapping the points of a plane π onto the points of another plane ω by perspection from an exterior point A – is not even defined if π and ω are affine. We see from Figure 1 that just as the points on the line $\boldsymbol{h}$ appear out of nowhere, the points on the line $\boldsymbol{k}$ have nowhere to go! Excluding these strange lines, the *painter's map* is quite well behaved. In fact, it is bijective:

$$\pi\backslash\{\boldsymbol{k}\} \hookrightarrow\!\!\!\!\rightarrow \omega\backslash\{\boldsymbol{h}\}$$

Projectifying π and ω clears up everything and gives us a bijection:

$$\pi \hookrightarrow\!\!\!\!\rightarrow \omega$$

The line $\boldsymbol{k}$ in π goes into $\boldsymbol{q}_\infty$ (the line at infinity) in ω, and the line $\boldsymbol{h}$ in ω comes from $\boldsymbol{p}_\infty$ (the line at infinity) in π.

Observe how $\boldsymbol{p}_\infty$ and $\boldsymbol{q}_\infty$ correspond precisely to the planes through A that are parallel to π and ω respectively, while all the other planes through A cut both π and ω in affine lines.

The affine planes through A thus correspond bijectively to the lines of the projective plane π, and in the same way the affine lines through A correspond bijectively to the points of the projective plane π.

Note that this correspondence is such that the line of intersection of two planes through A corresponds to the point of intersection of the corresponding lines of π. Hence, the points and lines of the projective plane can be modelled by the lines and planes through a fixed point in affine 3-space with their usual relations of incidence. We shall see in a moment how this fact can be used to introduce projective coordinates in $\mathbb{P}^2$.

Duality

The greatest advantage gained in projectifying the affine plane $\mathbf{E}^2$ is the fact that in the projective plane $\mathbf{P}^2$ the point and the line have a totally symmetric (or *dual*) relationship. Two elements of one kind determine one element of the other. In fact, each axiom of $\mathbf{P}^2$ has a dual counterpart, obtained by changing the word *point* into the word *line* and vice versa.

Duality is often indicated by writing dual statements in parallel columns e.g.

two different **points** are on one **line**	two different **lines** are on one **point**
the **points** on a **line** form a **range**	the **lines** on a **point** form a **pencil**

Since the axioms of $\mathbb{P}^2$ occur in dual pairs, it follows that the set of axioms as a whole is invariant under dualization. Hence each theorem has a dual counterpart, the proof of which can be obtained mechanically from a proof of the first part by dualizing each step in it. This pleasant fact is often referred to as **the principle of duality**.

Of course the principle of duality is not confined to $\mathbb{P}^2$ but is valid in any $\mathbb{P}^n$ with appropriate modification.

In $\mathbf{P}^3$ the **point** and the **plane** are dual elements:

three different **points** are on one **plane**	three different **planes** are on one **point**

while the **line** is self-dual, having two dual aspects (the **ray** and the **axis**):

two different **points** are on one **ray**	two different **planes** are on one **axis**
the **lines** on a **point** form a **star**	the **lines** on a **plane** form a **plane system**

How the duality principle in $\mathbb{P}^n$ creates a correspondence between its different *Grassmanian manifolds* is hinted at in Appendix 2.

Projective point coordinates

To introduce a projective coordinate system in $\mathbf{P}^2$ we choose any triplet of non-collinear points and call them the vertex 1, 2, 3 of the coordinate triangle (C.T.).

Then we choose any point U, not on any of the sides of the C.T. and call it *the unit point.*

Let us think of

$\mathbf{P}^2 = \{\pi\} \cup \{p_\infty\}$

as the affine plane π with its added line at infinity p_∞ and let π be embedded in $\mathbf{E}^3$ (Figure 2).

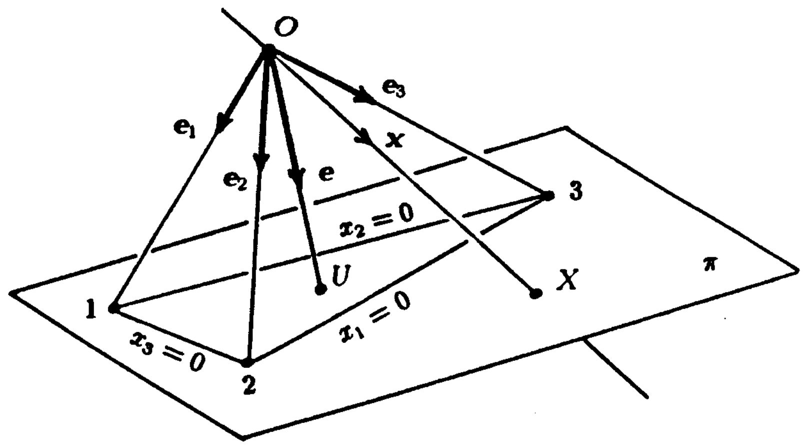


Figure 2

Choose a point O outside of π in $\mathbf{E}^3$ and vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ based at O and directed towards the points 1, 2, 3 in $\{\pi\} \cup \{p_\infty\}$, and adjust their lengths so that their sum

$$(1) \qquad \mathbf{e} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$$

is directed towards the unit point U.

As we indicated earlier $\mathbf{P}^2$ can be modelled by the non-zero subspaces of the linear space $\mathbf{E}^3_O$ of affine vectors based at O. Since the vertices 1, 2, 3 of the C.T. were chosen to be non-collinear, the chosen vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ form a basis of $\mathbf{E}^3_O$ called $\mathbf{B}$.

We now define the projective coordinates $[X]_{\text{C.T.U.}}$ of a point $X \in \mathbf{P}^2$, relative to the chosen Coordinate Triangle Unitpoint configuration: They are the ratios of the $\mathbf{B}$-coordinates

$$[\mathbf{x}]_\mathbf{B} = (x_1, x_2, x_3)$$

of any non-zero vector $\mathbf{x} \in \mathbf{E}^3_O$ on the line through O that intersects $\mathbf{P}^2$ in the point X. Hence

$$[X]_{\text{C.T.U.}} = (x_1 : x_2 : x_3) = (\rho x_1 : \rho x_2 : \rho x_3) \quad \text{whenever } \rho \neq 0.$$

Now, the projective coordinates of the point X are well defined, because they depend only on the choice of the C.T.U. system and not on the choice of the point O or the base $\mathbf{B}$ (as long as the sum $\rightarrow$ unitpoint condition (1) is fulfilled). The verification of this fact is an instructive exercise which is left to the interested reader.

From Figure 2 we immediately get

$$[1]_{\text{C.T.U.}} = (1:0:0)$$
$$[2]_{\text{C.T.U.}} = (0:1:0)$$
$$[3]_{\text{C.T.U.}} = (0:0:1)$$
$$[U]_{\text{C.T.U.}} = (1:1:1)$$

Any linear homogeneous equation

$$a^i x_i = 0 \quad \text{(tensor notation)}$$

describes a plane on O in $\mathbf{E}^3_O$ and hence a line of $\mathbb{P}^2$. Note that the sides of the C.T. are described by the planes $x_i = 0, \quad i = 1,2,3$.

If the points A and B of $\mathbb{P}^2$ are represented by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ of $\mathbf{E}^3_O$, we see that as λ and μ vary over $\mathbb{R}$, the plane $\lambda\boldsymbol{a} + \mu\boldsymbol{b}$ (as a family of lines in $\mathbf{E}^3_O$) represents the line AB (as a family of points in $\mathbb{P}^2$). These points thus have projective coordinates

$$((\lambda a_1 + \mu b_1) : (\lambda a_2 + \mu b_2) : (\lambda a_3 + \mu b_3)) \tag{2}$$

the ratio $(\lambda : \mu)$ being fixed for each point.

Note that we must exclude the value $(\lambda, \mu) = (0,0)$ since it does not give us a line in $\mathbf{E}^3_O$.

Hence $(\lambda : \mu)$ are projective coordinates of a point X in $\mathbb{P}^1$, and changing the names of the basevectors $\boldsymbol{a}$ and $\boldsymbol{b}$ to $\mathbf{e}_1$ and $\mathbf{e}_2$ in accordance with Figure 2 we get the situation of Figure 3.

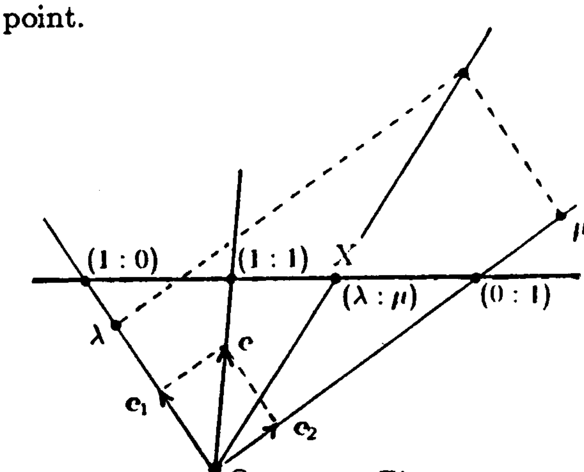


Figure 3

Returning to $\mathbb{P}^2$, let us consider it again as

$$\mathbb{P}^2 = \{\pi\} \cup \{p_\infty\}$$

with π embedded in $\mathbf{E}^3$ (Figure 2). By a suitable choice of the C.T.U. system, we obtain a correspondence between projective coordinates of $\mathbb{P}^2$ and cartesian coordinates of π (Figure 4).

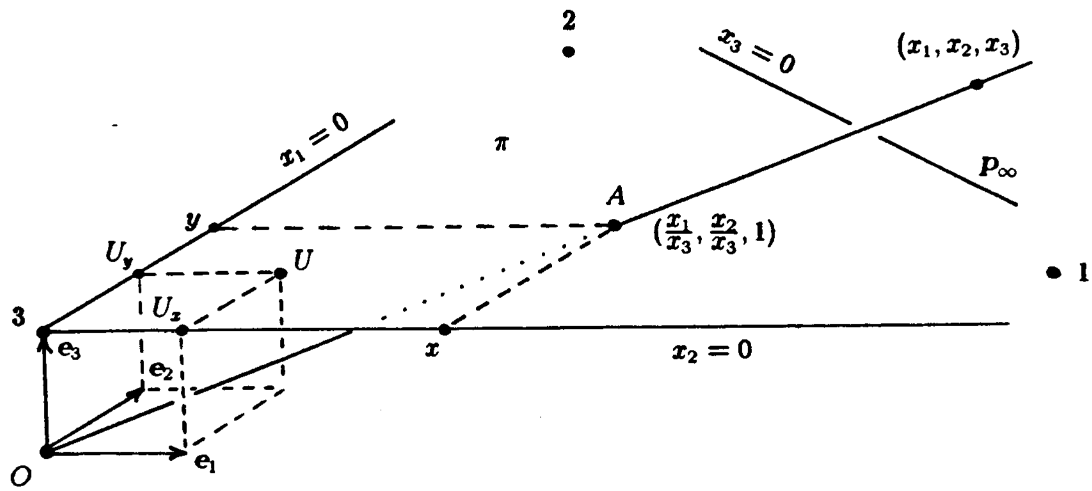


Figure 4

Here B= $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an ON-basis for $\mathbf{E}^3_O$, and the affine plane π is parallel to the affine plane $x_3 = 0$, determined by $\mathbf{e}_1$ and $\mathbf{e}_2$. These two planes intersect in the line p_∞ which therefore has the equation $x_3 = 0$ in projective coordinates.

Note that p_∞ contains the vertices 1 and 2 of the C.T. while vertex 3 becomes the origin of the corresponding cartesian coordinate system in π (called π-cart). Note also how the unit point U projects from vertex 2 and 1 of the C.T. to create the units U_x and U_y on the cartesian $x-$ and $y-$axes. Let (x_1, x_2, x_3) be the B-coordinates of a non-zero vector in $\mathbf{E}^3_O$. The affine line on O that it determines will intersect the affine plane π if and only if $x_3 \neq 0$. Assuming this to be the case, the point of intersection A will clearly satisfy

$$
\begin{aligned}
[A]_{\text{C.T.U.}} &= (x_1 : x_2 : x_3) \\
[A]_{\mathbf{B}} &= (\frac{x_1}{x_3}, \frac{x_2}{x_3}, 1) \\
[A]_{\pi-\text{cart.}} &= (\frac{x_1}{x_3}, \frac{x_2}{x_3})
\end{aligned}
\tag{3}
$$

Conversely, starting with a cartesian coordinate system in π, we can always obtain projective coordinates in $\{\pi\} \cup \{p_\infty\}$ by interpreting the cartesian system as part of a projective C.T.U. system as in Figure 4.

Algebraically this amounts to introducing new variables x_1, x_2, x_3 by letting

$$x = \frac{x_1}{x_3} \quad \text{and} \quad y = \frac{x_2}{x_3}$$

leading again to (3).

Of course, the analogous constructions can be carried out in any $\mathbb{P}^n$ to obtain a projective point coordinate system.

In $\mathbf{P}^3$ we get the following correspondences:

(4)

$\mathbf{P}^3$	$\mathbf{E}^4_O$
points	lines on O
lines	planes on O
planes	3-spaces on O

Hence, given any three non-collinear points X, Y, Z in $\mathbb{P}^3$, the plane they determine consists of all points with projective coordinates:

$$\lambda x_i + \mu y_i + \vartheta z_i \quad , \quad i = 1, \ldots, 4 \tag{5}$$

the ratio $(\lambda : \mu : \vartheta)$ being fixed for each point and $(0 : 0 : 0)$ being excluded.

In $\mathbb{P}^3$ the C.T. is of course a tetrahedron (Figure 5) and the unit point U is any chosen point not on any of its sides.

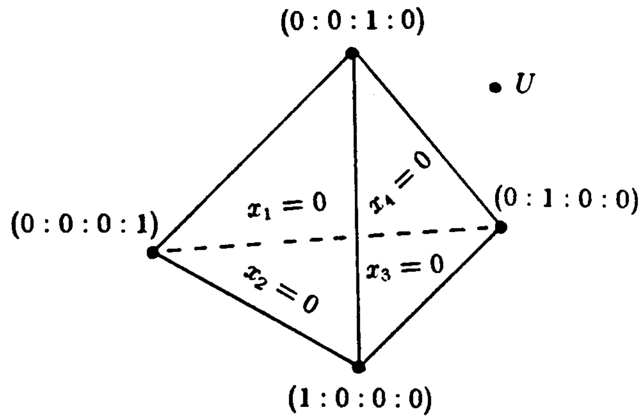
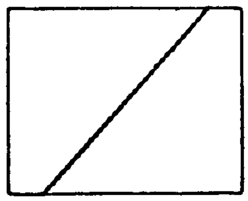


Figure 5

Algebraic aspects of duality

It was Julius Plücker who first dualized the cartesian plane algebraically, and treated points and lines on an equal basis. He observed that there are two symmetric and equally valid points of view:



If u and v are **constants**
and x and y are **variables**
then
$ux + vy + 1 = 0$
is the equation
of the **line**(u, v)
in **point** coordinates
(6) excluding
the **pencil of lines**
on the **origin**
of the coordinate system
and the **line at infinity**.

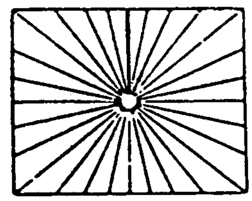
If the **points**
(x_1, y_1) and (x_2, y_2)
are on the **line**
$ax + by + c = 0$
we must have
$ax_1 + by_1 + c = 0$
$ax_2 + by_2 + c = 0$

The condition
that these 3 equations
be consistent

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

is the equation
of the **line**
joining the two **points**.

~ · ~



If u and v are **variables**
and x and y are **constants**
then
$ux + vy + 1 = 0$
is the equation
of the **point**(x, y)
in **line** coordinates
excluding
the **range of points**
on the **line at infinity**
of the coordinate system
and the **point at the origin**.

If the **lines**
(u_1, v_1) and (u_2, v_2)
are on the **point**
$au + bv + c = 0$
we must have
$au_1 + bv_1 + c = 0$
$au_2 + bv_2 + c = 0$

The condition
that these 3 equations
be consistent

$$\begin{vmatrix} u & v & 1 \\ u_1 & v_1 & 1 \\ u_2 & v_2 & 1 \end{vmatrix} = 0$$

is the equation
of the **point**
joining the two **lines**.

~ · ~

Projectifying ($x := x/z,\quad y := y/z$) to the natural C.T.U. extension of the cartesian coordinate system (Figure 4), the exceptions (6) disappear, and the **point/line** duality in $\mathbf{P}^2$ becomes complete:

If u, v, w are **constants**
and x, y, z are **variables**
then
$ux + vy + wz = 0$
is the equation
of the **line**$(u : v : w)$
in **point** coordinates
with no exception at all.

If the **points**
$(x_1 : y_1 : z_1)$ and $(x_2 : y_2 : z_2)$
are on the **line**
$ax + by + cz = 0$
we must have
$ax_1 + by_1 + cz_1 = 0$
$ax_2 + by_2 + cz_2 = 0$

The condition
that these 3 equations
be consistent

$$\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0$$

is the equation of the **line**
joining the two **points**

∼ · ∼

If u, v, w are **variables**
and x, y, z are **constants**
then
$ux + vy + wz = 0$
is the equation
of the **point**$(x : y : z)$
in **line** coordinates
with no exception at all.

If the **lines**
$(u_1 : v_1 : w_1)$ and $(u_2 : v_2 : w_2)$
are on the **point**
$au + bv + cw = 0$
we must have
$au_1 + bv_1 + cw_1 = 0$
$au_2 + bv_2 + cw_2 = 0$

The condition
that these 3 equations
be consistent

$$\begin{vmatrix} u & v & w \\ u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \end{vmatrix} = 0$$

is the equation of the **point**
joining the two **lines**

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(7) $$ux + vy + wz = 0$$

is called *the incidence relation*
of point and line

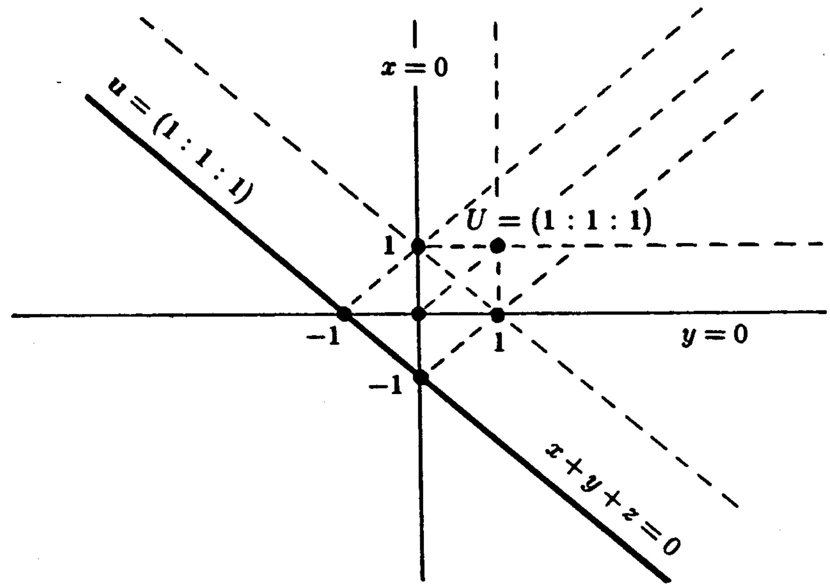


Figure 6

Let us call the C.T.U. system of Figure 4 a Projectified Cartesian System (P.C.S.). The relation between the unit point U and the unit line $\mathbf{u}$ of a P.C.S. in $\mathbb{P}^2$ is shown in Figure 6.

This is a special case of a relation called **trilinear pole/polar** between a point and a line in $\mathbb{P}^2$ with respect to a selected triangle.

Trilinear polarity and its counterpart in $\mathbb{P}^n$ (T-linear polarity) are discussed in Appendix 1.

Projective hyperplane coordinates

Dualizing the C.T.U. system for points in $\mathbf{P}^n$ we get the c.t.u system for *dual-points* in $\mathbf{P}^n$. Dual-points (or hyperplanes) are the *missing-1* dimensional linear objects of $\mathbf{P}^n$.

(8) In $\mathbf{P}^n$, the C.T. is of course a $(n+1)$-point simplex. Since each of its points avoids exactly one of its hyperplanes, we can dualize it into itself by mapping each point (vertex) onto its avoiding hyperplane and vice versa.

This turns our C.T. into a c.t. in a natural way that is especially well suited to handle duality algebraically. But this is not enough to ensure the bilinear type of incidence relation (7) between point and hyperplane, which creates complete algebraic duality.

(9) For this to happen, we must in addition choose the unit point U and the unit hyperplane $\mathbf{u}$ in a T-linear pole/polar position relative to the C.T. (Appendix 1).

Let us agree to call a C.T.U. system and a c.t.u. system **dually unified** if they are related according to (8) and (9). Hence we can state the following fundamental fact:

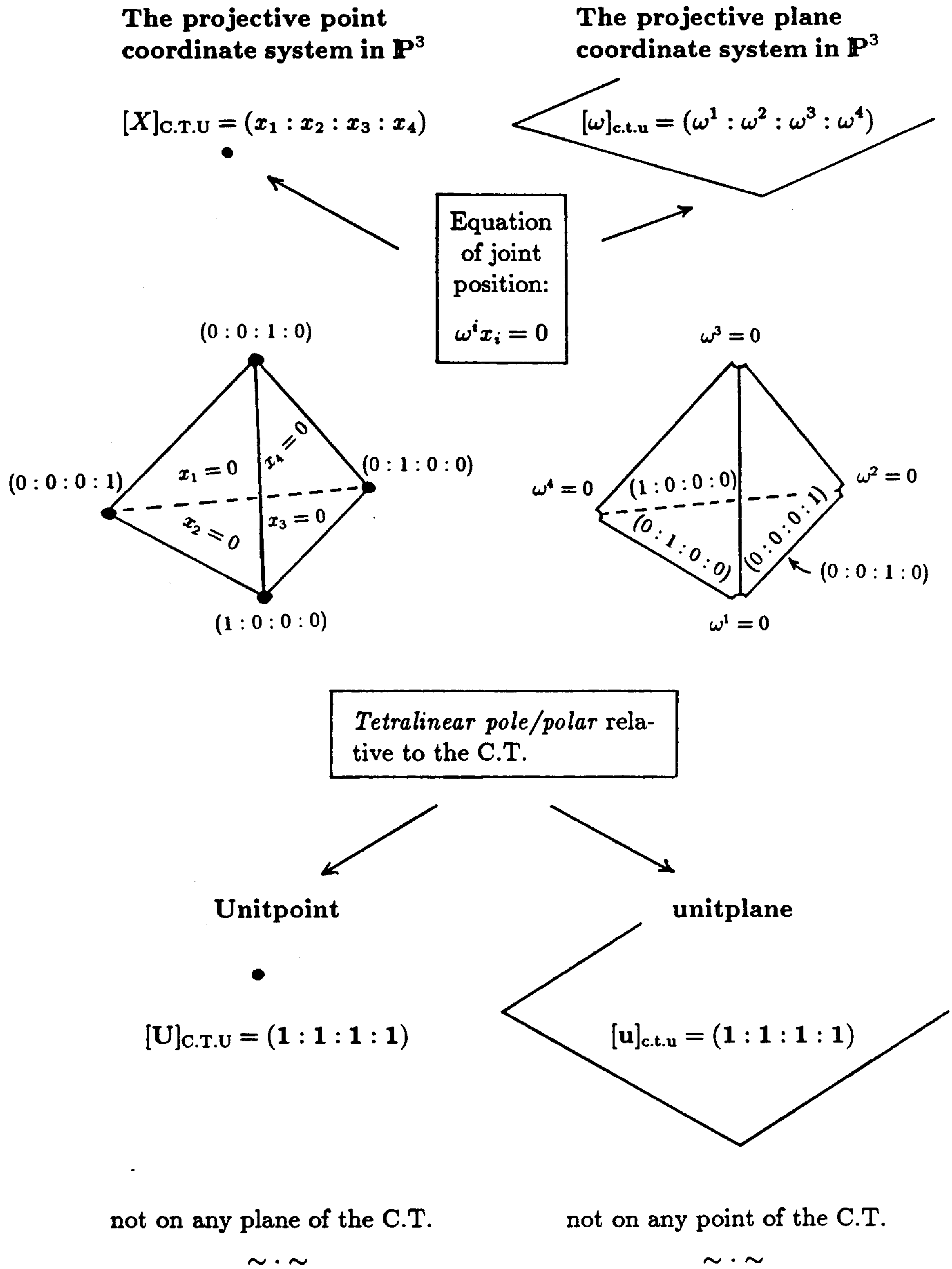
If X and ω are a point and a hyperplane in $\mathbf{P}^n$ and if

$$[X]_{\text{C.T.U.}} = (x_1 : x_2 : \ldots : x_{n+1})$$
$$[\omega]_{\text{c.t.u.}} = (\omega^1 : \omega^2 : \ldots : \omega^{n+1})$$

are their projective coordinates in two dually unified systems, then X and ω are incident if and only if

$$\omega^i x_i = \omega^1 x_1 + \omega^2 x_2 + \ldots + \omega^{n+1} x_{n+1} = 0 \qquad (10)$$

The proof this fact can be found in [2].



The projective point coordinate system in $\mathbb{P}^3$
$[X]_{C.T.U} = (x_1 : x_2 : x_3 : x_4)$
The projective plane coordinate system in $\mathbb{P}^3$
$[\omega]_{c.t.u} = (\omega^1 : \omega^2 : \omega^3 : \omega^4)$
Equation of joint position:
$\omega^i x_i = 0$
(0 : 0 : 1 : 0)
(0 : 0 : 0 : 1)
(0 : 1 : 0 : 0)
(1 : 0 : 0 : 0)
$x_1 = 0$
$x_2 = 0$
$x_3 = 0$
$x_4 = 0$
$\omega^3 = 0$
$\omega^4 = 0$
$\omega^2 = 0$
$\omega^1 = 0$
(1 : 0 : 0 : 0)
(0 : 1 : 0 : 0)
(0 : 0 : 0 : 1)
(0 : 0 : 1 : 0)
Tetralinear pole/polar relative to the C.T.
Unitpoint
unitplane
$[\mathbf{U}]_{C.T.U} = (\mathbf{1} : \mathbf{1} : \mathbf{1} : \mathbf{1})$
$[\mathbf{u}]_{c.t.u} = (\mathbf{1} : \mathbf{1} : \mathbf{1} : \mathbf{1})$
not on any plane of the C.T.
not on any point of the C.T.

Figure 7

Collineations

Having thus established a coordinate system for $\mathbb{P}^3$ we are now ready to study maps of $\mathbf{P}^3$ into itself.

Given a non-singular 4×4 real matrix (α_i^k)
a map

$$\mathbf{P}^3 \ni X \longmapsto Y \in \mathbf{P}^3$$

can be constructed by letting

$$y_i = \alpha_i^k x_k \quad , \quad i = 1, \ldots, 4 \tag{11}$$

Such a map is called a *collineation.*

The following two statements are fundamental facts of projective geometry in $\mathbf{P}^3$:

(i) There is exactly one collineation that maps any 5 generic* points onto any other 5 such points.

(ii) Every map $\mathbb{P}^3 \to \mathbb{P}^3$ which bijectively relates points $\leftrightarrow$ points and lines $\leftrightarrow$ lines is a collineation.

Note that (i) implies that each change of coordinate system is given by a unique collineation. Hence (11) can be interpreted either

as a change of image on a fixed background } **alibi** viewpoint

or

as a change of background on a fixed image } **alias** viewpoint

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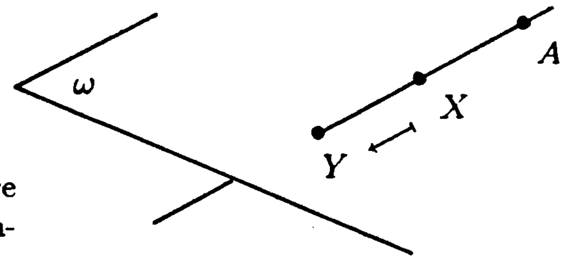
* no 4 of which are in the same plane

The visual operator in point coordinates

Let us now consider the perspective transformation S



(12) $\mathbf{P}^3 \backslash \{A\} \ni X \overset{S}{\longmapsto} Y \in \omega$

All points of the line AX exept A are mapped onto Y which is the point of intersection of AX with the plane ω.

Since Y is on the line AX, by (2) its coordinates can be expressed as

$$(13) \qquad y_i = \lambda a_i + \mu x_i \quad , \quad i = 1, \ldots, 4$$

for some real numbers λ and μ with fixed ratio. (Note that since $A \notin \omega$ we must have $\mu \neq 0$).

Moreover, since $Y \in \omega$ we get

$$(14) \qquad \omega^i y_i = 0$$

Substituting (13) into (14) gives

$$(15) \qquad \omega^i(\lambda a_i + \mu x_i) = 0$$

Since $A \notin \omega$ implies $\omega^i a_i \neq 0$ we can solve (15) for λ:

$$(16) \qquad \lambda = -\frac{\omega^i x_i}{\omega^k a_k}\mu$$

Plugging (16) into (13) we get

$$(17) \qquad \frac{\omega^k a_k}{\mu} y_i = (-\omega^j a_i + \omega^k a_k \delta_i^j) x_j \quad , \quad i = 1, \ldots, 4$$

where δ_i^j is the Kronecker delta function.

Letting $\beta_i = \sum_{j \neq i} \omega^j a_j$ we finally get in matrix notation

$$(18) \qquad \rho \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} -\beta_1 & \omega^2 a_1 & \omega^3 a_1 & \omega^4 a_1 \\ \omega^1 a_2 & -\beta_2 & \omega^3 a_2 & \omega^4 a_2 \\ \omega^1 a_3 & \omega^2 a_3 & -\beta_3 & \omega^4 a_3 \\ \omega^1 a_4 & \omega^2 a_4 & \omega^3 a_4 & -\beta_4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

where $\rho = -\omega^k a_k / \mu$ is a non-zero real factor of proportionality.

Denoting the 4×4 matrix in (18) by $Ç = (c_i^k)$ we have thus found a matrix representation for S. Let us examine its structure a bit closer.

Since the coordinates of X and Y in (18) are homogenous, the matrix $\lambda Ç$ will of course also represent S for any real $\lambda \neq 0$. If Y is any point in ω, we have $\omega^i y_i = 0$ and a simple calculation using this fact gives

$$Ç \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = -\omega^i a_i \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \tag{19}$$

Hence (y_i) is an eigenvector of $Ç$ with non-zero eigenvalue $-\omega^i a_i$, which reflects the fact that $S(Y) = Y$ for any point $Y \in \omega$.

Since ω contains 3 non-collinear points (whose coordinate vectors by (2) are linearly independent) $Ç$ has 3 linearly independent eigenvectors with non-zero eigenvalue. Hence we must have

$$\text{rank } Ç \geq 3 \tag{20}$$

But

$$Ç \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{21}$$

which together whith (20) means that

$$\ker Ç = \{\lambda(a_i) : \lambda \in \mathbb{R}\} \tag{22}$$

Hence ker $Ç$ corresponds to the *forbidden point* A and we have

$$Ç(\lambda(a_i) + \mu(x_i)) = \lambda Ç(a_i) + \mu Ç(x_i) = \mu Ç(x_i)$$

reflecting the fact that

$$S(\lambda A + \mu X) = S(X)$$

Also, it is easily verified that

$$Ç^2 = -\omega^i a_i Ç \tag{23}$$

Hence $Ç$ is *projectively idempotent* reflecting the fact that

$$S^2 = S$$

Let us recall the expression (11) for a collineation:

$\mathbb{P}^3 \ni X \longmapsto Y \in \mathbb{P}^3$

$$y_i = \alpha_i^k x_k \quad , \quad |\alpha_i^k| \neq 0$$

Comparing this to the expression (18) for a perspective transformation:

$\mathbf{P}^3\backslash\{A\} \ni X \longmapsto Y \in \omega \subset \mathbf{P}^3$

$$y_i = c_i^k x_k \quad , \quad |c_i^k| = 0$$

we see that it is natural to call the latter map **a singular collineation**, especially since its nullspace by (22) corresponds to its restricted domain.

Notation: The perspective transformation $\mathbf{P}^3\backslash\{A\} \xrightarrow{S} \omega$ will be called S_A^ω and its representative matrix (18) will be called $Ç_A^\omega$.

Fact: For each point $A = (a_i)_1^4$, $a_i \neq 0\ \forall i$, there are 4 canonical projections onto the 4 planes of the C.T.

Using (18) we can express their matrices:

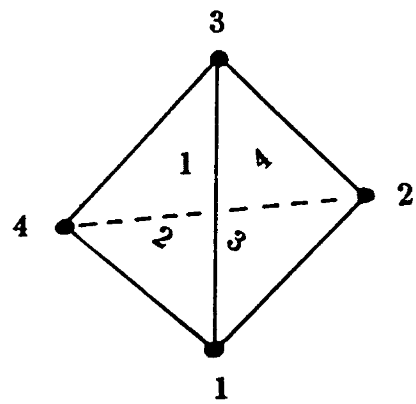
$$Ç_A^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_2 & -a_1 & 0 & 0 \\ a_3 & 0 & -a_1 & 0 \\ a_4 & 0 & 0 & -a_1 \end{pmatrix}$$

$$Ç_A^2 = \begin{pmatrix} -a_2 & a_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a_3 & -a_2 & 0 \\ 0 & a_4 & 0 & -a_2 \end{pmatrix}$$

(24)

$$Ç_A^3 = \begin{pmatrix} -a_3 & 0 & a_1 & 0 \\ 0 & -a_3 & a_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_4 & -a_3 \end{pmatrix}$$

$$Ç_A^4 = \begin{pmatrix} -a_4 & 0 & 0 & a_1 \\ 0 & -a_4 & 0 & a_2 \\ 0 & 0 & -a_4 & a_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$



Line geometry

Having thus developed a projective point representation of the perspective transformation S_A^ω, let us remind ourselves that we want to use it to map lines. Hence we would like to have a representation of this operator that tells us immediately what happens to a line $\boldsymbol{p}$ in $\mathbf{P}^3$ when it is mapped by S_A^ω , i.e. which gives us directly the position of the image line $S_A^\omega(\boldsymbol{p})$ in the plane ω without having to compute it by first mapping two points on $\boldsymbol{p}$.

In order to achieve this we must abandon our treatment of lines as collections of points and start to look at them as basic elements of our space. This was the idea of Julius Plücker as presented in his book *Neue Geometrie des Raumes gegründet auf die Betrachtung der geraden Linie als Raumelement (1868).* This subject has come to be known as **line geometry**. Before making use of it we will present a brief overview of the elementary aspects of projective line geometry of $\mathbb{P}^3$.

Ray coordinates of lines in $\mathbb{P}^3$

A line $\boldsymbol{p}$ in $\mathbf{P}^3$ is determined by two different points X and X' with projective point coordinates $(x_i)_1^4$ and $(x'_i)_1^4$.

From the six 2×2 minors of the matrix

$$\begin{pmatrix} x'_1 & x'_2 & x'_3 & x'_4 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix} \tag{25}$$

we define, for the line $\boldsymbol{p}$ its six homogeneous projective ray coordinates:

$$\begin{cases} \sigma p_1 = x'_1 x_4 - x'_4 x_1 \quad , \quad \sigma p_2 = x'_2 x_4 - x'_4 x_2 \quad , \quad \sigma p_3 = x'_3 x_4 - x'_4 x_3 \\ \sigma p_4 = x_2 x'_3 - x_3 x'_2 \quad , \quad \sigma p_5 = x_3 x'_1 - x_1 x'_3 \quad , \quad \sigma p_6 = x_1 x'_2 - x_2 x'_1 \end{cases} \tag{26}$$

where σ is an arbitrary non-zero real factor. These coordinates are denoted $(p_\rho)_1^6$ or simply (p_ρ) since from now on all latin indices will range from 1 to 4 and all greek indices from 1 to 6.

Consider the determinant identity

$$\begin{vmatrix} x'_1 & x'_2 & x'_3 & x'_4 \\ x_1 & x_2 & x_3 & x_4 \\ x'_1 & x'_2 & x'_3 & x'_4 \\ x_1 & x_2 & x_3 & x_4 \end{vmatrix} = 0$$

Laplace expansion of the left hand side gives the following identity for the ray coordinates:

$$p_1 p_4 + p_2 p_5 + p_3 p_6 = 0 \tag{27}$$

The ray coordinates of **p** are independent of the position of the two points X and X' along **p** (as long as they don't coincide) because (cfr (2))

$$(28) \qquad \left.\begin{aligned} y_i &= \alpha x_i + \beta x'_i \\ y'_k &= \alpha' x_k + \beta' x'_k \end{aligned}\right\} \quad \text{with} \quad d = \begin{vmatrix} \alpha\alpha' \\ \beta\beta' \end{vmatrix} \neq 0$$

gives

$$y'_i y_k - y'_k y_i = d \cdot (x'_i x_k - x'_k x_i)$$

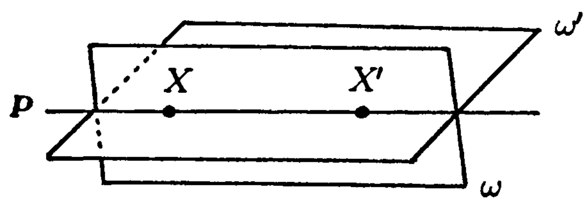
Axis coordinates of lines in $\mathbb{P}^3$

The line **p** is determined also by two different planes ω and ω' with projective plane coordinates (ω^i) and (ω'^i). The ray coordinates that were defined in (26) can then be expressed as

$$(29) \quad \left\{\begin{array}{lll} \lambda p_1 = \omega^2\omega'^3 - \omega^3\omega'^2 \;, & \lambda p_2 = \omega^3\omega'^1 - \omega^1\omega'^3 \;, & \lambda p_3 = \omega^1\omega'^2 - \omega^2\omega'^1 \\ \lambda p_4 = \omega'^1\omega^4 - \omega'^4\omega^1 \;, & \lambda p_5 = \omega'^2\omega^4 - \omega'^4\omega^2 \;, & \lambda p_6 = \omega'^3\omega^4 - \omega'^4\omega^3 \end{array}\right.$$

(λ arbitrary real $\neq 0$)

This follows from the equations of joint position (10) between the points X and X' and the planes ω and ω'.



If we define (cfr(26)) for a line **p** the homogenous projective axis coordinates (π^ρ):

$$(30) \quad \left\{\begin{array}{lll} \sigma\pi^1 = \omega'^1\omega^4 - \omega'^4\omega^1 \;, & \sigma\pi^2 = \omega'^2\omega^4 - \omega'^4\omega^2 \;, & \sigma\pi^3 = \omega'^3\omega^4 - \omega'^4\omega^3 \\ \sigma\pi^4 = \omega^2\omega'^3 - \omega^3\omega'^2 \;, & \sigma\pi^5 = \omega^3\omega'^1 - \omega^1\omega'^3 \;, & \sigma\pi^6 = \omega^1\omega'^2 - \omega^2\omega'^1 \end{array}\right.$$

then, comparing (29) and (30) we have for the same line **p** :

$$(31) \qquad (p_1 : p_2 : p_3 : p_4 : p_5 : p_6) = (\pi^4 : \pi^5 : \pi^6 : \pi^1 : \pi^2 : \pi^3)$$

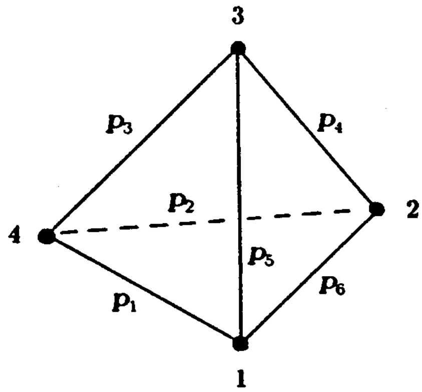
This can also be expressed as

$$(32) \qquad \pi^\rho = p_{\rho+3 \pmod 6}$$

Figure 8 illustrates the duality between the ray and the axis systems of line coordinates.

From the projective point coordinate system we construct

The ray coordinates for lines in $\mathbf{P}^3$



The edges of the C.T. have the coordinates

$$p_\mu = (0 : 0 : \ldots : 1 : \ldots : 0)$$

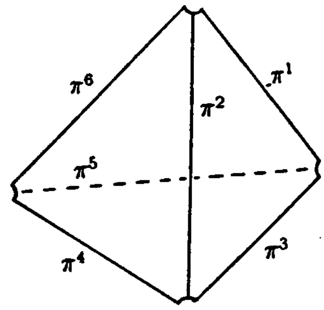
position μ

$$\mu = 1, 2, \ldots, 6$$

$\sim \cdot \sim$

From the projective plane coordinate system we construct

The axis coordinates for lines in $\mathbf{P}^3$



The edges of the C.T. have the coordinates

$$\pi^\mu = (0 : 0 : \ldots : 1 : \ldots : 0)$$

position μ

$$\mu = 1, 2, \ldots, 6$$

$\sim \cdot \sim$

Figure 8

The relations of a point and a plane to a line in $\mathbb{P}^3$

The line $p = (p_\rho)$ and the point $Z = (z_i)$ determine the plane $\omega = (\omega^i)$ with coordinates

The line $p = (\pi^\rho)$ and the plane $\omega = (\omega^i)$ determine the point $Z = (z_i)$ with coordinates

(33)
$$\begin{pmatrix}\omega^1\\ \omega^2\\ \omega^3\\ \omega^4\end{pmatrix} = \begin{pmatrix} 0 & -p_3 & p_2 & p_4\\ p_3 & 0 & -p_1 & p_5\\ -p_2 & p_1 & 0 & p_6\\ -p_4 & -p_5 & -p_6 & 0\end{pmatrix}\begin{pmatrix}z_1\\ z_2\\ z_3\\ z_4\end{pmatrix} \qquad \begin{pmatrix}z_1\\ z_2\\ z_3\\ z_4\end{pmatrix} = \begin{pmatrix} 0 & -\pi^3 & \pi^2 & \pi^4\\ \pi^3 & 0 & -\pi^1 & \pi^5\\ -\pi^2 & \pi^1 & 0 & \pi^6\\ -\pi^4 & -\pi^5 & -\pi^6 & 0\end{pmatrix}\begin{pmatrix}\omega^1\\ \omega^2\\ \omega^3\\ \omega^4\end{pmatrix}$$

unless
the product is zero
in which case
the line p contains
the point Z

~·~

unless
the product is zero
in which case
the line p is contained in
the plane ω

~·~

Proof: (We will prove the left part. The right part is the dual statement.)

Let $X = (x_i)$ and $X' = (x'_i)$ be two different points on the line p, and let $Y = (y_i)$ be an arbitrary point in $\mathbf{P}^3$. By (5) Y is in the plane ω (determined by points Z, X, X') if and only if

$$\begin{vmatrix} y_1 & y_2 & y_3 & y_4\\ z_1 & z_2 & z_3 & z_4\\ x_1 & x_2 & x_3 & x_4\\ x'_1 & x'_2 & x'_3 & x'_4\end{vmatrix} = 0$$

By (10) the plane coordinates of ω can be calculated by expansion along the first row of this determinant. We get

$$(\omega^i) = \left(+\begin{vmatrix} z_2 & z_3 & z_4\\ x_2 & x_3 & x_4\\ x'_2 & x'_3 & x'_4\end{vmatrix} : -\begin{vmatrix} z_1 & z_3 & z_4\\ x_1 & x_3 & x_4\\ x'_1 & x'_3 & x'_4\end{vmatrix} : +\begin{vmatrix} z_1 & z_2 & z_4\\ x_1 & x_2 & x_4\\ x'_1 & x'_2 & x'_4\end{vmatrix} : -\begin{vmatrix} z_1 & z_2 & z_3\\ x_1 & x_2 & x_3\\ x'_1 & x'_2 & x'_3\end{vmatrix}\right)$$

Expanding each of these four determinants along the first row and making use of (26) gives (33).

Finally, ω is undetermined $\Leftrightarrow$ all ω^i are zero $\Leftrightarrow$ the line p contains the point Z.

The intersection of two lines in $\mathbb{P}^3$

Given two lines $\boldsymbol{p} = (p_\rho)$ and $\boldsymbol{q} = (q_\rho)$ we define

(34) $$\boldsymbol{pq} = p_1q_4 + p_2q_5 + p_3q_6 + p_4q_1 + p_5q_2 + p_6q_3$$

Claim: Two different lines $\boldsymbol{p}$ and $\boldsymbol{q}$ intersect if and only if $\boldsymbol{pq} = 0$

Proof: Let X, X' and Y, Y' be two pairs of points on the lines $\boldsymbol{p}$ and $\boldsymbol{q}$ respectively. Then the intersection exists if and only if the determinant $|X, X', Y, Y'| = 0$. Laplace expansion along columns one and two gives (34).

If the lines $\boldsymbol{p}$ and $\boldsymbol{q}$ intersect, we have from (33):

Their common point:

(36) $$((q_6p_5 - q_5p_6) : (q_4p_6 - q_6p_4) : (q_5p_4 - q_4p_5) : (q_1p_4 + q_2p_5 + q_3p_6))$$

Their common plane:

(37) $$((q_3p_2 - q_2p_3) : (q_1p_3 - q_3p_1) : (q_2p_1 - q_1p_2) : (q_4p_1 + q_5p_2 + q_6p_3))$$

Note: If we permute the vertices of the coordinate tetrahedron (Figure 8) and renumber the ray coordinates p_ρ and q_ρ accordingly, we get permuted versions of the formulas (36) and (37). It may happen that the *original* versions of these formulas produce the forbidden result $(0 : 0 : 0 : 0)$ when applied to two intersecting lines. This nuicance can always be eliminated by a suitable permutation.

Collineations in ray coordinates

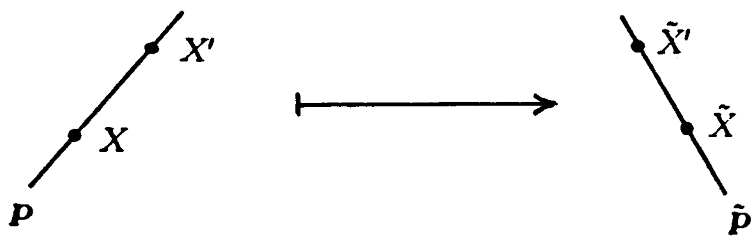
Returning again to the collineation (11)

$$\mathbf{P}^3 \ni X \longmapsto \tilde{X} \in \mathbf{P}^3$$

we recall its expression in point coordinates

(38) $$\tilde{x}_i = \alpha_i^k x_k \quad , \quad |\alpha_i^k| \neq 0$$

Since it maps a line $\boldsymbol{p}$ to a line $\tilde{\boldsymbol{p}}$



we can express its action on lines in ray coordinates.

From (26) we have for the image line $\tilde{\boldsymbol{p}}$:

(39) $$\sigma\tilde{p}_1 = \tilde{x}_1'\tilde{x}_4 - \tilde{x}_4'\tilde{x}_1 \quad , \quad \sigma\tilde{p}_2 = \dots \quad , \quad \sigma\tilde{p}_3 = \dots \quad , \quad \dots$$

Plugging (38) into (39) we get

$$\begin{aligned}
\sigma\tilde{p}_1 &= (\alpha_1^k x_k')(\alpha_4^j x_j) - (\alpha_4^k x_k')(\alpha_1^j x_j) = \\
&= (\alpha_1^1 x_1' + \alpha_1^2 x_2' + \alpha_1^3 x_3' + \alpha_1^4 x_4')(\alpha_4^1 x_1 + \alpha_4^2 x_2 + \alpha_4^3 x_3 + \alpha_4^4 x_4) - \\
&\quad - (\alpha_4^1 x_1' + \alpha_4^2 x_2' + \alpha_4^3 x_3' + \alpha_4^4 x_4')(\alpha_1^1 x_1 + \alpha_1^2 x_2 + \alpha_1^3 x_3 + \alpha_1^4 x_4) = \\
&= (\alpha_1^1\alpha_4^4 - \alpha_1^4\alpha_4^1)x_1'x_4 - (\alpha_1^1\alpha_4^4 - \alpha_1^4\alpha_4^1)x_4'x_1 + \\
&\quad + (\alpha_1^2\alpha_4^4 - \alpha_1^4\alpha_4^2)x_2'x_4 - (\alpha_1^2\alpha_4^4 - \alpha_1^4\alpha_4^2)x_4'x_2 + \\
&\quad + (\alpha_1^3\alpha_4^4 - \alpha_1^4\alpha_4^3)x_3'x_4 - (\alpha_1^3\alpha_4^4 - \alpha_1^4\alpha_4^3)x_4'x_3 + \\
&\quad + (\alpha_1^3\alpha_4^2 - \alpha_1^2\alpha_4^3)x_2x_3' - (\alpha_1^3\alpha_4^2 - \alpha_1^2\alpha_4^3)x_3x_2' + \\
&\quad + (\alpha_1^1\alpha_4^3 - \alpha_1^3\alpha_4^1)x_3x_1' - (\alpha_1^1\alpha_4^3 - \alpha_1^3\alpha_4^1)x_1x_3' + \\
&\quad + (\alpha_1^2\alpha_4^1 - \alpha_1^1\alpha_4^2)x_1x_2' - (\alpha_1^2\alpha_4^1 - \alpha_1^1\alpha_4^2)x_2x_1' = \\
&= \gamma_1^1 p_1 + \gamma_1^2 p_2 + \gamma_1^3 p_3 + \gamma_1^4 p_4 + \gamma_1^5 p_5 + \gamma_1^6 p_6
\end{aligned} \qquad (40)$$

In the same way we get the corresponding expressions for $\sigma\tilde{p}_2, \dots, \sigma\tilde{p}_6$ and we have the following expression for a collineation (38) in ray coordinates:

(41) $$\tilde{p}_\rho = \gamma_\rho^\mu p_\mu \quad , \quad (\rho = 1, \dots, 6)$$

The 36 coefficients γ_ρ^μ in (41) are determined by the 16 coefficients α_i^k in (38) according to the following rule:

$$
(42)\qquad \left\{\begin{array}{ll}
\gamma_1^1 = \alpha_1^1\alpha_4^4 - \alpha_1^4\alpha_4^1 \quad , \quad \ldots & \gamma_4^1 = \alpha_3^1\alpha_2^4 - \alpha_3^4\alpha_2^1 \quad , \quad \ldots \\
\gamma_1^2 = ---------- & \gamma_4^2 = ---------- \\
\gamma_1^3 = ---------- & \gamma_4^3 = ---------- \\
\gamma_1^4 = \alpha_1^3\alpha_4^2 - \alpha_1^2\alpha_4^3 \quad , \quad \ldots & \gamma_4^4 = \alpha_2^2\alpha_3^3 - \alpha_2^3\alpha_3^2 \quad , \quad \ldots \\
\gamma_1^5 = ---------- & \gamma_4^5 = ---------- \\
\gamma_1^6 = ---------- & \gamma_4^6 = ----------
\end{array}\right.
$$

The dotted expressions (...) follow from the ones to the left by the cyclic permutation of the lower index 1, 2, 3 while keeping fixed all the index 4.

The dashed expressions (—) follow from the ones above by the cyclic permutation of the upper index 1, 2, 3 while keeping fixed all the index 4.

Since we will make use of it later, we have taken the trouble to write out (42) explicitely in Table 1.

Table 1: γ_ρ^μ as a function of α_i^k

$\gamma_1^1 = \alpha_1^1\alpha_4^4 - \alpha_1^4\alpha_4^1$	$\gamma_2^1 = \alpha_2^1\alpha_4^4 - \alpha_2^4\alpha_4^1$	$\gamma_3^1 = \alpha_3^1\alpha_4^4 - \alpha_3^4\alpha_4^1$
$\gamma_1^2 = \alpha_1^2\alpha_4^4 - \alpha_1^4\alpha_4^2$	$\gamma_2^2 = \alpha_2^2\alpha_4^4 - \alpha_2^4\alpha_4^2$	$\gamma_3^2 = \alpha_3^2\alpha_4^4 - \alpha_3^4\alpha_4^2$
$\gamma_1^3 = \alpha_1^3\alpha_4^4 - \alpha_1^4\alpha_4^3$	$\gamma_2^3 = \alpha_2^3\alpha_4^4 - \alpha_2^4\alpha_4^3$	$\gamma_3^3 = \alpha_3^3\alpha_4^4 - \alpha_3^4\alpha_4^3$
$\gamma_1^4 = \alpha_1^3\alpha_4^2 - \alpha_1^2\alpha_4^3$	$\gamma_2^4 = \alpha_2^3\alpha_4^2 - \alpha_2^2\alpha_4^3$	$\gamma_3^4 = \alpha_3^3\alpha_4^2 - \alpha_3^2\alpha_4^3$
$\gamma_1^5 = \alpha_1^1\alpha_4^3 - \alpha_1^3\alpha_4^1$	$\gamma_2^5 = \alpha_2^1\alpha_4^3 - \alpha_2^3\alpha_4^1$	$\gamma_3^5 = \alpha_3^1\alpha_4^3 - \alpha_3^3\alpha_4^1$
$\gamma_1^6 = \alpha_1^2\alpha_4^1 - \alpha_1^1\alpha_4^2$	$\gamma_2^6 = \alpha_2^2\alpha_4^1 - \alpha_2^1\alpha_4^2$	$\gamma_3^6 = \alpha_3^2\alpha_4^1 - \alpha_3^1\alpha_4^2$
$\gamma_4^1 = \alpha_3^1\alpha_2^4 - \alpha_3^4\alpha_2^1$	$\gamma_5^1 = \alpha_1^1\alpha_3^4 - \alpha_1^4\alpha_3^1$	$\gamma_6^1 = \alpha_2^1\alpha_1^4 - \alpha_2^4\alpha_1^1$
$\gamma_4^2 = \alpha_3^2\alpha_2^4 - \alpha_3^4\alpha_2^2$	$\gamma_5^2 = \alpha_1^2\alpha_3^4 - \alpha_1^4\alpha_3^2$	$\gamma_6^2 = \alpha_2^2\alpha_1^4 - \alpha_2^4\alpha_1^2$
$\gamma_4^3 = \alpha_3^3\alpha_2^4 - \alpha_3^4\alpha_2^3$	$\gamma_5^3 = \alpha_1^3\alpha_3^4 - \alpha_1^4\alpha_3^3$	$\gamma_6^3 = \alpha_2^3\alpha_1^4 - \alpha_2^4\alpha_1^3$
$\gamma_4^4 = \alpha_2^2\alpha_3^3 - \alpha_2^3\alpha_3^2$	$\gamma_5^4 = \alpha_3^2\alpha_1^3 - \alpha_3^3\alpha_1^2$	$\gamma_6^4 = \alpha_1^2\alpha_2^3 - \alpha_1^3\alpha_2^2$
$\gamma_4^5 = \alpha_2^3\alpha_3^1 - \alpha_2^1\alpha_3^3$	$\gamma_5^5 = \alpha_3^3\alpha_1^1 - \alpha_3^1\alpha_1^3$	$\gamma_6^5 = \alpha_1^3\alpha_2^1 - \alpha_1^1\alpha_2^3$
$\gamma_4^6 = \alpha_2^1\alpha_3^2 - \alpha_2^2\alpha_3^1$	$\gamma_5^6 = \alpha_3^1\alpha_1^2 - \alpha_3^2\alpha_1^1$	$\gamma_6^6 = \alpha_1^1\alpha_2^2 - \alpha_1^2\alpha_2^1$

Sixvectors

A sixvector $\boldsymbol{p}$ is by definition a sixtuple $(p_\rho)_1^6$ of real numbers that is transformed by a collineation according to (41).

Given two sixvectors $\boldsymbol{p}$ and $\boldsymbol{q}$, we define their product as in (34):

(43) $$\boldsymbol{pq} = p_1q_4 + p_2q_5 + p_3q_6 + p_4q_1 + p_5q_2 + p_6q_3$$

A sixvector $\boldsymbol{p}$ is called:

singular if $\boldsymbol{pp} = 0$
non-singular if $\boldsymbol{pp} \neq 0$

By selecting a coordinate system for lines in $\mathbb{P}^3$, one can establish the following 3 facts:

(44) (*i*) every line in $\mathbb{P}^3$ gives a non-zero singular sixvector (through its line coordinates in the selected system)

(45) (*ii*) every non-zero singular sixvector gives a line in $\mathbb{P}^3$ (by interpreting it as line coordinates in the selected system)

(46) (*iii*) two non-zero singular sixvectors give the same line if and only if they are linearly dependent

Statement (*i*) is an immediate consequence of (26), (27) and (41). The proof of statements (*ii*) and (*iii*) is straightforward, but we omit it here. The reader is referred to [8] ch. 1.

The following operative rules for sixvectors are easily established ($\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{r}$ sixvectors, λ real constant):

(47) $$\boldsymbol{p} + \boldsymbol{q} = \boldsymbol{q} + \boldsymbol{p} \quad ; \quad \lambda(\boldsymbol{p} + \boldsymbol{q}) = \lambda\boldsymbol{p} + \lambda\boldsymbol{q}$$
$$\boldsymbol{pq} = \boldsymbol{qp} \quad ; \quad \boldsymbol{p}(\boldsymbol{q} + \boldsymbol{r}) = \boldsymbol{pq} + \boldsymbol{pr}$$

Linear families of lines in $\mathbb{P}^3$

Consider a point X and a plane ω in $\mathbf{P}^3$. The set of lines on X is called a **star** and the set of lines in ω is called a **plane system**. If $X \in \omega$ the set of lines on X and in ω is called a **pencil**.

By (45) and (46) two linearly independent singular sixvectors $\boldsymbol{p}$ and $\boldsymbol{q}$ with $\boldsymbol{pq} = 0$ determine two intersecting lines and thus a pencil. It is easily seen that this pencil can be expressed as:

$$\{\lambda_1\boldsymbol{p} + \lambda_2\boldsymbol{q} \mid (\lambda_1, \lambda_2) \neq (0,0)\} \tag{48}$$

(Note that $(\lambda_1\boldsymbol{p} + \lambda_2\boldsymbol{q})(\lambda_1\boldsymbol{p} + \lambda_2\boldsymbol{q}) = 0$)

In the same way three linearly independent singular sixvectors $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{r}$ with $\boldsymbol{pq} = \boldsymbol{qr} = \boldsymbol{rp} = 0$ determine three pairwise intersecting lines in $\mathbf{P}^3$ that do not belong to the same pencil (since their sixvectors are linearly independent).

Hence

if $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{r}$ are on the same point they determine a star of lines given by:

if $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{r}$ are in the same plane they determine a plane system of lines given by:

$$\{\lambda_1\boldsymbol{p} + \lambda_2\boldsymbol{q} + \lambda_3\boldsymbol{r} \quad \mid \quad (\lambda_1, \lambda_2, \lambda_3) \neq (0,0,0)\} \tag{49}$$

The visual operator in ray coordinates

Having completed our brief introduction to projective line geometry, we are now in a position to apply it to our study of the visual operator, viz. the perspective transformation

$$S_A^\omega : \mathbf{P}^3 \backslash \{A\} \longrightarrow \omega \quad , \quad A \notin \omega$$

First of all let us observe that in changing the representation of a collineation (38) from point — to ray coordinates, our derivation of the transformation formula (42) never made use of the fact that the collineation was non-singular. Hence (42) must hold also for singular collineations such as S_A^ω, and we can therefore use it to determine the representation of this operator in ray coordinates.

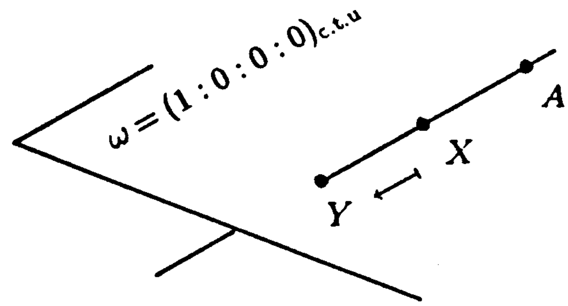
Secondly, in doing so there is no loss of generality to assume that the coordinate tetrahedron (Figure 7) has been chosen to contain ω as one of its planes, say $\omega = (1:0:0:0)_{\text{c.t.u.}}$

This will help us to avoid unnecessary computational complexity and make the structure of the resulting representation more clear.

Consider therefore the perspective transformation

$$S_A^1 : \mathbf{P}^3 \backslash \{A\} \rightarrow \omega$$

By (24) it is represented in point coordinates by the matrix



$$\varsigma_A^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_2 & -a_1 & 0 & 0 \\ a_3 & 0 & -a_1 & 0 \\ a_4 & 0 & 0 & -a_1 \end{pmatrix} = (\alpha_i^k)$$

Note: $A \notin \omega \Rightarrow a_1 \neq 0$

From (42) or Table 1 we determine its representation in ray coordinates:

(50)

$\gamma_1^1 = 0$	$\gamma_2^1 = -a_1a_2$	$\gamma_3^1 = -a_1a_3$
$\gamma_1^2 = 0$	$\gamma_2^2 = a_1^2$	$\gamma_3^2 = 0$
$\gamma_1^3 = 0$	$\gamma_2^3 = 0$	$\gamma_3^3 = a_1^2$
$\gamma_1^4 = 0$	$\gamma_2^4 = 0$	$\gamma_3^4 = 0$
$\gamma_1^5 = 0$	$\gamma_2^5 = 0$	$\gamma_3^5 = a_1a_4$
$\gamma_1^6 = 0$	$\gamma_2^6 = -a_1a_4$	$\gamma_3^6 = 0$
$\gamma_4^1 = 0$	$\gamma_5^1 = 0$	$\gamma_6^1 = 0$
$\gamma_4^2 = 0$	$\gamma_5^2 = 0$	$\gamma_6^2 = 0$
$\gamma_4^3 = 0$	$\gamma_5^3 = 0$	$\gamma_6^3 = 0$
$\gamma_4^4 = a_1^2$	$\gamma_5^4 = 0$	$\gamma_6^4 = 0$
$\gamma_4^5 = a_1a_2$	$\gamma_5^5 = 0$	$\gamma_6^5 = 0$
$\gamma_4^6 = a_1a_3$	$\gamma_5^6 = 0$	$\gamma_6^6 = 0$

In matrix form we get:

$$\widehat{\c{C}}_A^1 = \gamma_\rho^\mu = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -a_1a_2 & a_1^2 & 0 & 0 & 0 & -a_1a_4 \\ -a_1a_3 & 0 & a_1^2 & 0 & a_1a_4 & 0 \\ 0 & 0 & 0 & a_1^2 & a_1a_2 & a_1a_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{51}$$

Comparing with (41) we see that the matrix $\widehat{\c{C}}_A^1$ represents S_A^1 the following way: Given a line $\boldsymbol{p} = (p_\rho)$, the ray coordinates (q_ρ) of the image line $\boldsymbol{q} = S_A^1(\boldsymbol{p})$ are given by the matrix product

$$q_\rho = \gamma_\rho^\mu p_\mu \tag{52}$$

Having thus arrived at the line geometric description of the visual operator promised at the outset of this paper, let us examine its algebraic structure to see how it reflects the geometric properties of the perspective tranformation.

First, since $a_1 \neq 0$ (otherwise A would lie in ω and S_A^ω would not exist) we see from (51) that $\widehat{\c{C}}_A^1$ has rank = 3. Hence by the dimension theorem of linear algebra

$$\dim(\ker \widehat{\c{C}}_A^1) = 3 \tag{53}$$

Geometrically it is obvious that S_A^ω maps a line $\boldsymbol{p}$ to a line $S_A^\omega(\boldsymbol{p})$ unless the line $\boldsymbol{p}$ happens to pass through A (in which case it is mapped to a point). Hence from (44) and (45) we see that the star of lines on A must correspond exactly to the kernel of the matrix $\widehat{\c{C}}_A^1$. This is in agreement with (53) since by (49) a star of lines has linear dimension 3.

To verify this algebraically, take an arbitrary point $B \neq A$ and let $\boldsymbol{p}$ be the line AB. From (26) we get its ray coordinates

$$(54) \qquad \begin{cases} p_1 = b_1a_4 - b_4a_1 \ , \ p_2 = b_2a_4 - b_4a_2 \ , \ p_3 = b_3a_4 - b_4a_3 \\ p_4 = a_2b_3 - a_3b_2 \ , \ p_5 = a_3b_1 - a_1b_3 \ , \ p_6 = a_1b_2 - a_2b_1 \end{cases}$$

and one verifies that $\widehat{\c{C}}^1_A(p_\rho) = 0$ as we anticipated.

While we are multiplying matrices, we might as well note that

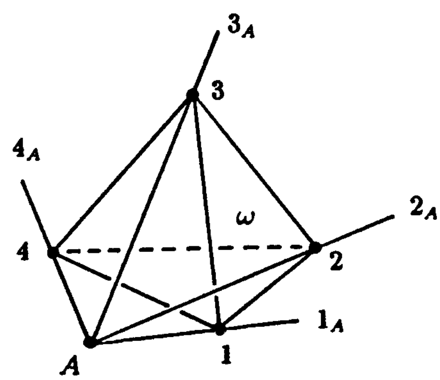
$$(\widehat{\c{C}}^1_A)^2 = a_1^2 \widehat{\c{C}}^1_A$$

reflecting the fact that S^1_A is idempotent.

There is an illuminating way to describe the kernel of $\widehat{\c{C}}^1_A$:
By(54) its general member is

$$\begin{pmatrix} b_1a_4 - b_4a_1 \\ b_2a_4 - b_4a_2 \\ b_3a_4 - b_4a_3 \\ a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \equiv b_1 \underbrace{\begin{pmatrix} a_4 \\ 0 \\ 0 \\ 0 \\ a_3 \\ -a_2 \end{pmatrix}}_{1_A} + b_2 \underbrace{\begin{pmatrix} 0 \\ a_4 \\ 0 \\ -a_3 \\ 0 \\ a_1 \end{pmatrix}}_{2_A} + b_3 \underbrace{\begin{pmatrix} 0 \\ 0 \\ a_4 \\ a_2 \\ -a_1 \\ 0 \end{pmatrix}}_{3_A} + b_4 \underbrace{\begin{pmatrix} -a_1 \\ -a_2 \\ -a_3 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{4_A}$$

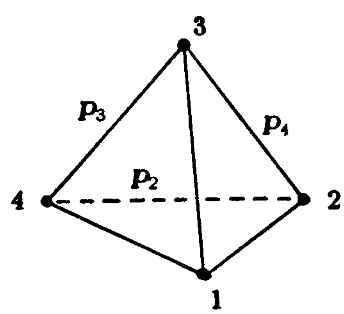
The sixvectors $1_A, 2_A, 3_A, 4_A$ are singular. Hence by (45) they are the ray coordinates of 4 different lines (if they are non-zero) and by (26) these are the lines that connect the point A to the 4 vertices of the C.T.



Since $a_1 \neq 0$ the sixvectors $2_A, 3_A, 4_A$ are always non-zero and linearly independent. Hence they form a basis for ker $\widehat{\c{C}}^1_A$. (Note that 1_A is zero in case $A =$ the point 1).

By (51) the image of $\widehat{\c{C}}^1_A$ is spanned by the sixvectors

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$



Comparing with Figure 8 we see that these are the edges $\boldsymbol{p}_2, \boldsymbol{p}_3, \boldsymbol{p}_4$ of the C.T. (which lie in the plane ω).

Hence the image of $\widehat{\c{C}}^1_A$ is the plane system of lines in ω as it should be.

Appendix 1: Tetralinear pole/polar in $\mathbb{P}^3$

A point and a plane with a certain relationship to a tetrahedron in $\mathbb{P}^3$ are said to be in a ***tetralinear pole/polar*** position relative to it. The point is called the ***tetralinear pole*** of the plane and the plane is called the ***tetralinear polar*** of the point with respect to the selected tetrahedron.

For the sake of clarity we will begin by describing the analogous relationship (trilinear pole/polar) between a point and a line in $\mathbb{P}^2$ with respect to a selected triangle (Figure 9).

Given a triangle ABC in $\mathbb{P}^2$ and a point Q not on any of its sides. We will construct a line $\mathbf{q}$ called the ***trilinear polar*** of Q with respect to the triangle ABC:

Join Q to A, B and C.

QA cuts BC in D,
QB cuts CA in F,
QC cuts AB in G.

Consider the intersections of corresponding sides of triangles ABC and DFG:

DF cuts AB in X,
FG cuts BC in Y,
GD cuts AC in Z.

It is easy to verify that X, Y and Z lie on one line.
This is the desired line $\mathbf{q}$.

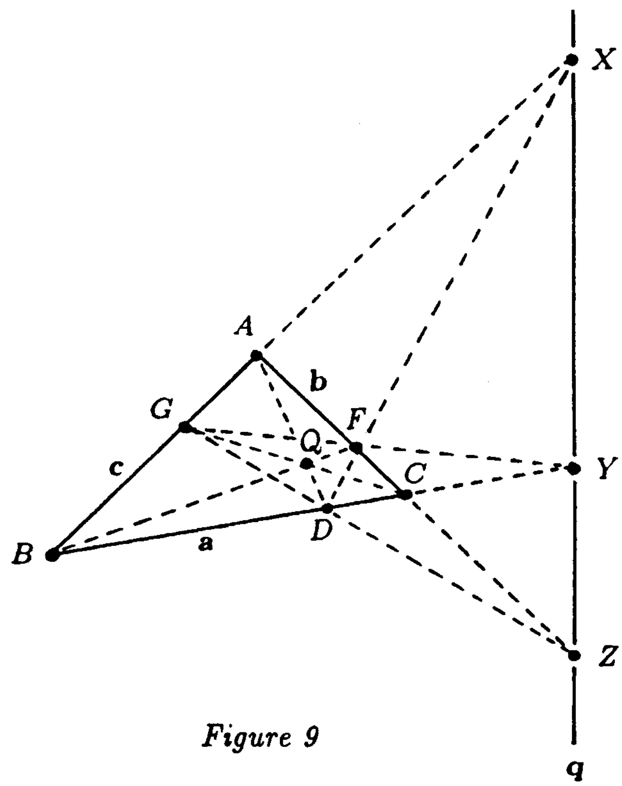


Figure 9

By starting with $\mathbf{q}$ and dualizing the above construction relative to the same triangle (now given by its sides $\mathbf{abc}$) we end up with 3 lines on a point. One verifies easily that this is the original point Q, therefore called the ***trilinear pole*** of $\mathbf{q}$ with respect to the triangle $\mathbf{abc}$.

It is instructive to consider the trilinear pole/polar relation with respect to the ordinary cartesian coordinate triangle (P.C.S): $x_1 = 0$ (the y-axis), $x_2 = 0$ (the x-axis) and $x_3 = 0$ (the line at infinity) (Figure 10).

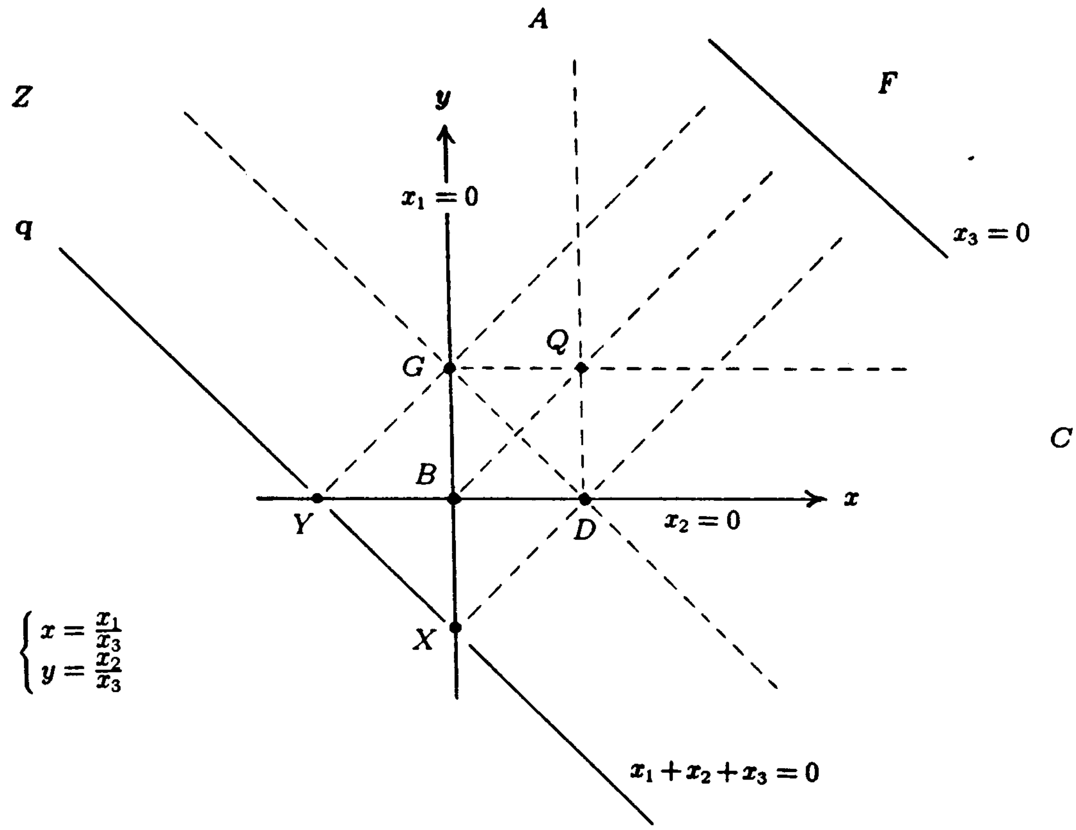


Figure 10

Given the unitpoint $Q = (1,1)_{\text{cart.}} = (1:1:1)_{\text{proj.}}$ the above construction gives the trilinear polar line $\boldsymbol{q}$ with cartesian equation $x + y + 1 = 0$ or projective equation $x_1 + x_2 + x_3 = 0$. Hence $\boldsymbol{q}$ has projective coordinates $(1 : 1 : 1)$. Note that the points A, C, F and Z lie at infinity.

The analogue of the trilinear pole/polar construction in $\mathbb{P}^2$ can be carried out in any $\mathbb{P}^n, n > 2$ in a natural way relative to a selected $(n+1)$-simplex. In $\mathbb{P}^3$ this results in the ***tetralinear pole/polar*** relationship of a point and a plane with respect to a tetrahedron that was mentioned in Figure 7.

If n is unspecified, we can refer to the corresponding relation between a point and a hyperplane in $\mathbb{P}^n$ as *T-linear polarity*.

Appendix 2: The Grassmanian coordinates in $\mathbb{P}^n$

In analogy with (4), $\mathbb{P}^n$ can be modelled as the set of all affine subspaces on a fixed point O in $\mathbb{E}^{n+1}$. The projective dimension of such a subspace is defined to be its affine dimension as a subspace of $\mathbb{E}^{n+1}$. Hence we have

element of $\mathbb{P}^n$:	point	line	plane	$\cdots$	$\mathbb{P}^n$
subspace of $\mathbb{E}_O^{n+1}$:	line	plane	3-space	$\cdots$	$(n+1)$-space
projective dimension :	1	2	3	$\cdots$	$(n+1)$

The set of all subspaces with projective dimension d is called the **d-Grassmannian** of $\mathbb{P}^n$, i.e.

$$\begin{aligned}
\{\text{points in } \mathbb{P}^n\} &= \text{the 1-Grassmanian of } \mathbb{P}^n \\
\{\text{lines in } \mathbb{P}^n\} &= \text{the 2-Grassmanian of } \mathbb{P}^n \\
\{\text{planes in } \mathbb{P}^n\} &= \text{the 3-Grassmanian of } \mathbb{P}^n \\
\vdots \quad & \quad \vdots \qquad\qquad \vdots
\end{aligned}$$

The d-Grassmannian of $\mathbb{P}^n$ can be coordinatized by the homogenous, non-zero $\binom{n+1}{d}$-tuples of d-minors of the $d \times (n+1)$ matrix:

$$\begin{pmatrix} x_1^1 & x_1^2 & \cdots & x_1^{n+1} \\ x_2^1 & x_2^2 & \cdots & x_2^{n+1} \\ \vdots & \vdots & & \vdots \\ x_d^1 & x_d^2 & \cdots & x_d^{n+1} \end{pmatrix} \tag{55}$$

where the rows are the coordinates of d generic points of one of its elements – a choice which guarantees that the rank of the matrix (55) is d – hence producing a non-zero *Grassmannian coordinate* $\binom{n+1}{d}$*-tuple* for each element.

It is important to observe that, in general, these coordinates are not independent but connected by certain equations. In the case of the 2-Grassmannian of $\mathbb{P}^3$, this connection is given by the *Plücker equation* (27), which is satisfied identically by all line coordinates.

The number of d-minors of (55), and the number of real dimensions (*degrees of freedom*) of the corresponding d-Grassmannians are shown in Figure 11.

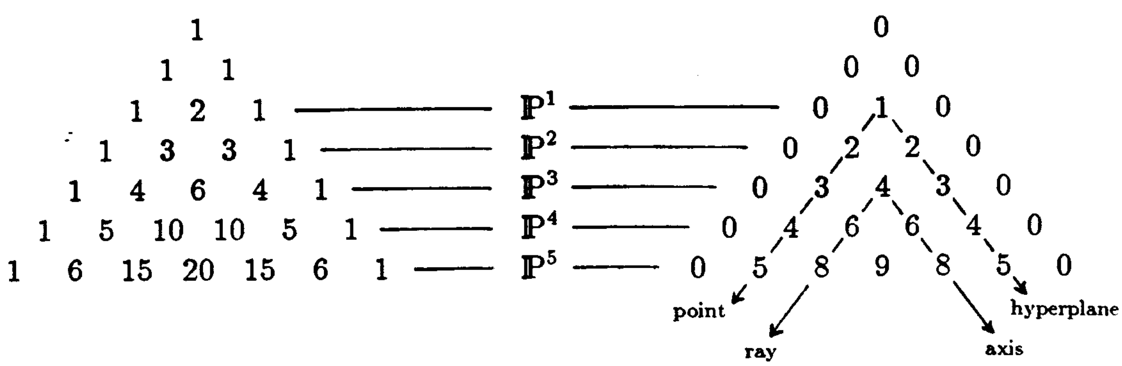


Figure 11

Since $\binom{n+1}{d} = \binom{n+1}{(n+1)-d}$, the d-Grassmannian and the $((n+1)-d)$-Grassmannian of $\mathbb{P}^n$ have the same dimension. In fact, the principle of duality in $\mathbb{P}^n$ makes them correspond bijectively as sets of dual elements.

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