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ON THE USE OF EXTERIOR ALGEBRA IN IMAGE ANALYSIS

by

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TRITA-NA-P8709

CVAP 30

Report from Computer Vision and Associative Pattern Processing Laboratory

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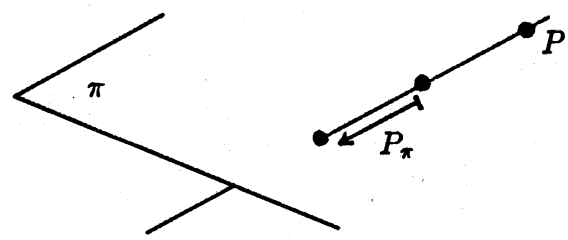
On the use of exterior algebra in image analysis

It is a well known (and annoying) fact that the perspective transformation



(1) $\qquad P_\pi : \mathbb{R}^3 \backslash \{P\} \to \pi \simeq \mathbb{R}^2$

is non-linear.

This difficulty can be handled in various ways, e.g. by introducing homogeneous coordinates in the usual manner, or by setting up a projective coordinate system and describing the perspective transformation projectively (see [8].)

These methods have an important drawback, they are coordinate dependent, i.e. they rely on the introduction of a specific coordinate system, which has to be chosen in a more or less ad hoc way. Of course this is tolerable in many cases where a coordinate system presents itself naturally e.g. in the analysis of a single image of a fixed object from a fixed (known) viewpoint. When motion phenomena are considered, however, the coordinate dependent descriptions often introduce considerable analytical complexity, and the benefits of a coordinate free description would be substantial.

It is a pleasant mathematical fact that there exists such a coordinate free description of the *linear part* of projective geometry, i.e. the points, lines, planes.... It is called the **exterior algebra** (or Grassmann algebra) and it is closely related to the algebra of subspaces of a finite dimensional vectorspace as we shall see in this paper.

Mathematical background

Let V be an n-dimensional vectorspace over $\mathbb{R}$. The Grassmann algebra over V, called $\wedge(V)$ is an associative algebra over $\mathbb{R}$ with the properties

(2) $\wedge(V)$ is a graded algebra, that is
$\wedge(V) = \wedge_0(V) \oplus \wedge_1(V) \oplus \wedge_2(V) \oplus \ldots \oplus \wedge_i(V) \oplus \ldots$
where each $\wedge_i(V)$ is a subspace of $\wedge(V)$ and for
$u \in \wedge_i(V)$, $v \in \wedge_j(V)$ we have $u \wedge v \in \wedge_{i+j}(V)$
where $\wedge$ denotes multiplication in $\wedge(V)$

(3) $\wedge_0(V) = \mathbb{R}$ and $\wedge_1(V) = V$

(4) $\wedge_1(V)$ together with the identity $1 \in \mathbb{R}$ generate $\wedge(V)$.

(5) $x \wedge x = 0 \quad \forall x \in \wedge_1(V)$

(6)
$$\left.\begin{array}{rcl} \rho x_1 \wedge \ldots \wedge x_n & = & 0 \\ x_1 \wedge \ldots \wedge x_n & \neq & 0 \\ x_1, \ldots, x_n & \in & \wedge_1(V) \end{array}\right\} \Longrightarrow \rho = 0$$

The properties (2), ..., (6) determine $\wedge(V)$ uniquely, i.e. any associative algebra over $\mathbb{R}$ satisfying them is isomorhic to $\wedge(V)$.

The Grassmann algebra $\wedge(V)$ can be realized as the quotient algebra $T(V)/I_e$, where $T(V)$ is the tensor algebra over V and I_e is the ideal generated by the elements of the form $x \otimes x$, $x \in V$.

The following properties of the algebra $\wedge(V)$ are easily established.

(7) $u \wedge v = (-1)^{ij} v \wedge u$, $u \in \wedge_i(V)$, $v \in \wedge_j(V)$

(8) If $\{e_1, \ldots, e_n\}$ is a basis of V, then $\{e_\psi\}$ is a basis of $\wedge(V)$, where ψ runs over all subsets of $\{1, \ldots, n\}$ including the empty set ; where $e_\psi = e_{i_1} \wedge \ldots \wedge e_{i_r}$ with $i_1 < \ldots < i_r$ when ψ is the subset $\{i_1, \ldots, i_r\}$ of $\{1, \ldots, n\}$; and where $e_\psi = 1$ when $\psi = \Phi$ (the empty set).

In particular
$\wedge_n(V) \cong \mathbb{R}$ and $\wedge_{n+j}(V) = \{0\}$, $j > 0$.

Moreover, it follows that

$\dim \wedge(V) = 2^n$
$\dim \wedge_k(V) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$ $,0 \leq k \leq n$

(9) q vectors $u_1, \ldots, u_q \in V$ are linearly independent if and only if $u_1 \wedge \ldots \wedge u_q \neq 0$

(10) If $u_1 = \sum a_{1k} v_k$, ... , $u_r = \sum a_{rk} v_k$
are r linear combinations of r vectors
v_1 , ... , $v_r \in V$, then
$u_1 \wedge \ldots \wedge u_r = \det(a_{jk})\, v_1 \wedge \ldots \wedge v_r$

An excellent account of the basic properties of the algebra $\wedge(V)$ can be found in Warner [1].

An element $u \in \wedge_p(V)$ is called **decomposable** if there are v_1 , ... , $v_p \in \wedge_1(V)$ such that $u = v_1 \wedge \ldots \wedge v_p$

Otherwise u is called indecomposable.

(11) If $\dim V \leq 3$ then every $u \in \wedge_p(V)$ is decomposable. If $\dim V > 3$ and $\{e_i\}$ is a basis of V then $e_1 \wedge e_2 + e_3 \wedge e_4$ is indecomposable.

(12) If $u \in \wedge_2(V)$, then u is decomposable if and only if $u \wedge u = 0$.

(13) All $u \in \wedge_{n-1}(V)$ are decomposable.

(14) If W is a p-dimensional subspace of V, then $\wedge_p(W)$ is a one-dimensional subspace of decomposable elements of $\wedge_p(V)$.

(15) If Y is a one-dimensional subspace of $\wedge_p(V)$ consisting only of decomposable elements, then $Y = \wedge_p(W)$ for some p-dimensional subspace W of V.

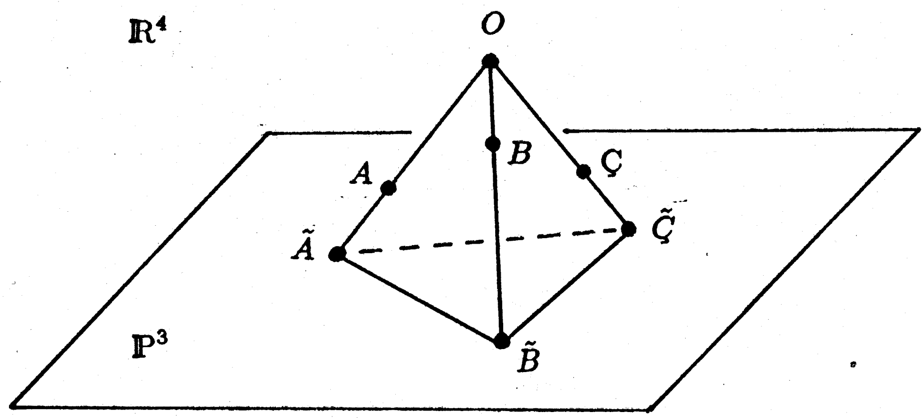
(16) Let W and X be subspaces of V of dimensions p and q, respectively, and let $\omega \in \wedge_p(W)$, $x \in \wedge_q(X)$, $\omega \neq 0$, $x \neq 0$

Then we have

(*i*) $X \subset W$ if and only if there is a decomposable y such that $\omega = x \wedge y$.
(*ii*) $X \cap W = 0$ if and only if $x \wedge \omega \neq 0$
(*iii*) If $X \cap W = 0$, then $\omega \wedge x$ is a basis of $\wedge_{p+q}(W + X)$
(*iv*) $W = \{v : v \in \wedge_1(V) \text{ and } v \wedge \omega = 0\}$.

In [8] it is described how the points, lines and planes of $\mathbb{P}^3$ (real projective 3-space) can be modelled by the 1-, 2-, and 3-dimensional subspaces of $\mathbb{R}^4$ respectively.

Let A, B and $Ç$ be points (vectors) of $\mathbb{R}^4$ different from the origin O, and not all in the same 2-d subspace. Then the three points determine three linearly independent 1-d subspaces OA, OB and $OÇ$ of $\mathbb{R}^4$, and hence three different non-collinear points $\tilde{A}, \tilde{B}$ and $\tilde{Ç}$ of $\mathbb{P}^3$.



Consider the two linearly independent vectors A and $\lambda A + \mu B$ ($\mu \neq 0$), spanning the 2-d subspace OAB of $\mathbb{R}^4$. Since $A \wedge A = 0$ we have

(17) $$A \wedge (\lambda A + \mu B) = \lambda A \wedge A + \mu A \wedge B = \mu A \wedge B.$$

Hence the subspace OAB is characterized by the element $A \wedge B \in \wedge_2(\mathbb{R}^4)$ modulo non-zero scaling. Introducing the equivalence class

(18) $$[A \wedge B] = \{\mu A \wedge B \ : \ \mu \neq 0\}$$

we have thus demonstrated that

(19) the 2-d subspace OAB of $\mathbb{R}^4$ $\longleftrightarrow$ $[A \wedge B]$
and hence:
the line $\tilde{A}\tilde{B}$ of $\mathbb{P}^3$ $\longleftrightarrow$ $[A \wedge B]$.

Considering in a similar way the 3-d subspace $OABÇ$ of $\mathbb{R}^4$, we get

(20) $$A \wedge B \wedge (\lambda A + \mu B + \nu Ç) = \nu A \wedge B \wedge Ç$$

where $\nu \neq 0$ since the three vectors A, B and $\lambda A + \mu B + \nu Ç$ are assumed to span $OABÇ$.

Hence we have

(21) the 3-d subspace $OABÇ$ of $\mathbb{R}^4$ $\longleftrightarrow$ $[A \wedge B \wedge Ç]$
and by translating this
to projective 3-space:
the plane $\tilde{A}\tilde{B}\tilde{Ç}$ of $\mathbb{P}^3$ $\longleftrightarrow$ $[A \wedge B \wedge Ç]$.

If we denote by $\wedge_i(V)/_\sim$ the set of all equivalence classes of decomposable elements in $\wedge_i(V)$ under the equivalence relation [ ], we can sum up these observations in the following correspondencies:

(22)
$$
\begin{aligned}
\wedge_0(\mathbf{R}^4)/_\sim &\longleftrightarrow \{1\} \cup \{0\} \\
\wedge_1(\mathbf{R}^4)/_\sim &\longleftrightarrow \{\text{points of } \mathbf{P}^3\} \cup \{0\} \\
\wedge_2(\mathbf{R}^4)/_\sim &\longleftrightarrow \{\text{lines of } \mathbf{P}^3\} \cup \{0\} \\
\wedge_3(\mathbf{R}^4)/_\sim &\longleftrightarrow \{\text{planes of } \mathbf{P}^3\} \cup \{0\} \\
\wedge_4(\mathbf{R}^4)/_\sim &\longleftrightarrow \mathbb{P}^3 \cup \{0\}
\end{aligned}
$$

Perspective transformation by multiplication in $\wedge(\mathbb{R}^4)/_\sim$

Let us now return to the perspective transformation (1). By projectifying the situation (see [8]) we get a map

$$\tilde{P}_\pi : \mathbf{P}^3 \backslash \{P\} \longrightarrow \mathbf{P}^2 \quad (23)$$

The reason for doing so is twofold:

First, the vanishing points become explicit and do not differ from any other points, and second, the transformation becomes linear when regarded as a mapping of subspaces in $\mathbb{R}^4$.

A minor nuisance is provided by the fact that a projective transformation corresponds to a whole class of equivalent linear (point) transformations, each one differing from any other by a non-zero multiplicative constant, reflecting the fact that homogeneous coordinates (as well as subspaces) are scale invariant.

Recalling that P is the point of $\mathbb{P}^3$ corresponding to the position of the eye, we have for any point $Q \neq P$, any line QR not on P and any plane QRS not on P:

$$
\begin{aligned}
Q &\longmapsto \text{the line } PQ \\
QR &\longmapsto \text{the plane } PQR \quad (24) \\
QRS &\longmapsto \text{the 3-space } PQRS
\end{aligned}
$$

Here we have adopted a *pre-retina* viewpoint, i.e. we have not yet intersected the geometric objects PQ, PQR and $PQRS$ with the image plane, thus keeping a record of all possible preimages.

Now, by (22), this version of the perspective transformation has a natural interpretation as a map

$$(25) \qquad \wedge(\mathbb{R}^4)/_\sim \xrightarrow{P\wedge[\cdot]} \wedge(\mathbb{R}^4)/_\sim$$

which could be called *wedging by the eye*
where

$$(26) \quad \begin{array}{llll} \wedge_0(\mathbb{R}^4)/_\sim & \ni \lambda & \longmapsto & [\lambda P] \in \wedge_1(\mathbb{R}^4)/_\sim \\ \wedge_1(\mathbb{R}^4)/_\sim & \ni [Q] & \longmapsto & [P\wedge Q] \in \wedge_2(\mathbb{R}^4)/_\sim \\ \wedge_2(\mathbb{R}^4)/_\sim & \ni [Q\wedge R] & \longmapsto & [P\wedge Q\wedge R] \in \wedge_3(\mathbb{R}^4)/_\sim \\ \wedge_3(\mathbb{R}^4)/_\sim & \ni [Q\wedge R\wedge S] & \longmapsto & [P\wedge Q\wedge R\wedge S] \in \wedge_4(\mathbb{R}^4)/_\sim \\ \wedge_4(\mathbb{R}^4)/_\sim & \ni [Q\wedge R\wedge S\wedge T] & \longmapsto & [P\wedge Q\wedge R\wedge S\wedge T] = 0 \end{array}$$

Note that $P\wedge[\cdot] = [P\wedge\cdot]$ for any element $\cdot$ of $\wedge_i(\mathbb{R}^4)$.

Also note that if any of the geometric objects Q, QR or QRS are on P, the corresponding elements in (26) are mapped to zero. Hence the map (25) also covers the projectively forbidden cases.

Grassmann coordinates

We have now seen how the Grassmann algebra on a vectorspace V allows us to perform coordinate free geometry on the subspaces of V and interpret the results in the corresponding projective space.

Let us denote by $S_k(V)$ the set of all k-dimensional subspaces of V. Suppose that we coordinatize V by introducing a basis $\mathcal{B} = \{e_1, \ldots, e_n\}$. This will induce a coordinatization of the elements of $S_k(V)$. The coordinates thus attached to a given element $[W] \in S_k(V)$ are called the Grassmann coordinates of $[W]$ and will be denoted by $[W]_\psi$.

To see how they are obtained, let us choose a basis $\mathcal{F} = \{f_1, \ldots, f_k\}$ for the subspace $[W] \in S_k(V)$. We then have

$$(27) \qquad f_i = \sum_{j=1}^{n} \lambda_{ji} e_j \quad , \quad i = 1, \ldots, k \quad , \quad \lambda_{ji} \in \mathbb{R}$$

By (5), (7) and (10) we can now write

$$(28) \qquad \begin{aligned} [W] &= [f_1 \wedge \ldots \wedge f_k] = [(\sum_{j_1} \lambda_{j_1 1} e_{j_1}) \wedge \ldots \wedge (\sum_{j_k} \lambda_{j_k k} e_{j_k}] = \\ &= [\sum_{j_1}^{n} \cdots \sum_{j_k}^{n} \lambda_{j_1 1} \cdot \ldots \cdot \lambda_{j_k k} e_{j_1} \wedge \ldots \wedge e_{j_k}] = \\ &= [\sum_{\substack{\psi \subset \{1,\ldots,n\} \\ \psi \text{ increasing} \\ |\psi| = k}} (\sum_{\{j_1,\ldots,j_k\} = \psi} (\mathrm{sgn}\ j) \lambda_{j_1 1} \cdot \ldots \cdot \lambda_{j_k k}) e_\psi] \end{aligned}$$

where the outer sum is over all selections ψ of k elements from $\{1,\ldots,n\}$ taken in increasing order, and the inner sum is over all permutations j of the elements $\{j_1,\ldots,j_k\}$ of a fixed ψ.

Now, it follows from (8) that

$$\{[e_\psi] : |\psi| = k\} \tag{29}$$

is a basis of $S_k(V)$, and hence we have obtained in (28) a coordinate expansion of $[W]$

$$[W] = \sum_{|\psi|=k} \lambda_\psi [e_\psi] \quad , \tag{30}$$

where

$$\lambda_\psi = \sum_{\{j_1,\ldots,j_k\}=\psi} (\operatorname{sgn} j) \lambda_{j_1 1} \cdot \ldots \cdot \lambda_{j_k k} \tag{31}$$

Hence the Grassmann coordinates of $[W]$ are

$$[W]_\psi = \{\lambda_\psi : |\psi| = k\}. \tag{32}$$

If we form the matrix

$$[\mathcal{F}]_B = \begin{pmatrix} | & & | \\ [f_1]_B & \cdots & [f_k]_B \\ | & & | \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \ldots & \lambda_{1k} \\ \vdots & & \\ \lambda_{n1} & \ldots & \lambda_{nk} \end{pmatrix} \tag{33}$$

we see that if $\psi = \{j_1 \,,\, \ldots, j_k\}$, $j_1 < \ldots < j_k$, λ_ψ is given by the $k \times k$ minor of the matrix $[\mathcal{F}]_B$ obtained by choosing the k rows $j_1,\ldots,j_k$ corresponding to ψ.

Note that the coordinates $\{\lambda_\psi\}$ are homogeneous, since by scaling the vectors f_i of $\mathcal{F}$, we can change the numbers $\{\lambda_\psi\}$ by any non-zero scaling factor.

In order to see that the Grassmann coordinates of $[W]$ are well defined, let us select another basis of $[W]$

$$\mathcal{F}' = \{f'_1,\ldots,f'_k\}. \tag{34}$$

We then have

$$f'_i = \sum_{\ell=1}^{n} \lambda'_{\ell i} e_\ell \quad , \quad i = 1,\ldots,k \tag{35}$$

and since both $\mathcal{F}$ and $\mathcal{F}'$ form a basis of $[W]$

$$f_i' = \sum_{j=1}^{k} a_{ji} f_j \quad , \quad \det\,(a_{ji}) \neq 0. \tag{36}$$

Substituting (27) into (36) we get

$$f_i' = \sum_{j=1}^{k} a_{ji} (\sum_{\ell=1}^{n} \lambda_{\ell j} e_\ell) = \sum_{\ell=1}^{n} (\sum_{j=1}^{k} \lambda_{\ell j} a_{ji}) e_\ell \tag{37}$$

and comparing (35) with (37) we have

$$\lambda_{\ell i}' = \sum_{j=1}^{k} \lambda_{\ell j} a_{ji} \tag{38}$$

or in matrix form, with $(a_{ji}) = A$

$$[\mathcal{F}']_{\mathcal{B}} = [\mathcal{F}]_{\mathcal{B}} \cdot A \tag{39}$$

Hence, the corresponding $k \times k$ minors of $[\mathcal{F}']_{\mathcal{B}}$ and $[\mathcal{F}]_{\mathcal{B}}$ are related by the same non-zero scaling factor $(\det A)$, i.e.

$$\lambda_\psi' = \lambda_\psi \cdot (\det A). \tag{40}$$

Therefore the $\binom{n}{k}$-tuples $\{\lambda_\psi'\}$ and $\{\lambda_\psi\}$ express the same homogeneous coordinate tuple and writing

$$[\lambda_\psi] = \{\alpha \lambda_\psi : \alpha \neq 0\} \tag{41}$$

we have

$$[W]_\psi = [\lambda_\psi] = [\lambda_\psi']. \tag{42}$$

Hence the Grassmann coordinates (32) of the subspace $[W]$ depend only on the choice of basis $\mathcal{B}$ for the entire space V, and not on the choice of basis $\mathcal{F}$, representing the subspace $[W]$.

Projective transformations of Grassmann coordinates

Let us start with a linear map

(43) $$T : V \rightarrow V$$

This map induces an algebra homomorphism

(44) $$\wedge T : \wedge(V) \rightarrow \wedge(V)$$

which on decomposable elements is defined by

(45) $$\wedge T(v_1 \wedge \ldots \wedge v_s) = Tv_1 \wedge \ldots \wedge Tv_s$$

and which is extended linearly.

The map $\wedge T$ in turn induces a map $[\wedge T]$ from the set $\wedge(V)/_\sim$ of all subspaces of V into itself

(46) $$\begin{aligned}[\wedge T] : \wedge(V)/_\sim &\rightarrow \wedge(V)/_\sim \\ [W] &\longmapsto [\wedge T(W)]\end{aligned}$$

What happens to our k-dimensional subspace $[W]$ under the mapping $[\wedge T]$, i.e. what are the Grassmann coordinates of the subspace $[\wedge T][W]$?

To answer this question we first pick a representative $W \in \wedge_k(V)$ of the subspace $[W]$ and study the action of $\wedge T$ on it.

$\wedge T$ is an algebra homomorphism of degree zero, hence a direct sum

(47) $$\wedge T = \bigoplus_k \wedge_k T \quad ,$$

where

(48) $$\wedge_k T : \wedge_k(V) \rightarrow \wedge_k(V)$$

is a linear map from the linear space $\wedge_k(V)$ into itself.

By (8), the basis $\mathcal{B} = \{e_1, \ldots, e_n\}$ of V induces a basis of $\wedge_k(V)$:

(49) $$\{e_\psi : \psi \subset \{1, \ldots, n\} \ , \ \psi \text{ increasing}, \ |\psi| = k\}$$

where, by convention $\psi = \{\psi_1, \ldots, \psi_k\}$ is assumed to be ordered increasingly, so that

(50) $$\psi_i < \psi_j \quad \text{if} \quad i < j.$$

To get the corresponding matrix representation of $\wedge_k T$, we must simply check its action on the basis elements e_ψ of $\wedge_k(V)$.

Assuming the matrix of the map T in the basis $\mathcal{B}$ to be $A = (a_{ij})$, we get

(51)
$$\begin{aligned}
\wedge_k T(e_\psi) &= \wedge_k T(e_{\psi_1} \wedge \ldots \wedge e_{\psi_k} = \\
&= Te_{\psi_1} \wedge \ldots \wedge Te_{\psi_k} = \\
&= (\sum_{i_1=1}^{n} a_{i_1\psi_1} e_{i_1}) \wedge \ldots \wedge (\sum_{i_k=1}^{n} a_{i_k\psi_k} e_{i_k}) = \\
&= \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} a_{i_1\psi_1} \cdot \ldots \cdot a_{i_k\psi_k} e_{i_1} \wedge \ldots \wedge e_{i_k} = \\
&= \sum_{\substack{\theta \subset \{1,\ldots,n\} \\ \theta \text{ increasing} \\ |\theta|=k}} (\sum_{\{i_1,\ldots,i_k\}=\theta} (\text{ sgn } i) a_{i_1\psi_1} \cdot \ldots \cdot a_{i_k\psi_k}) e_\theta = \\
&= \sum_{|\theta|=k} \underbrace{\begin{vmatrix} a_{\theta_1\psi_1} & \ldots & a_{\theta_1\psi_k} \\ \vdots & & \\ a_{\theta_k\psi_1} & \ldots & a_{\theta_k\psi_k} \end{vmatrix}}_{A_{\theta,\psi}} \cdot e_\theta
\end{aligned}$$

where θ is assumed to be ordered increasingly, just as ψ.

Hence we can summarize (51):

(52) $$\wedge_k T(e_\psi) = \sum_{|\theta|=k} A_{\theta,\psi} e_\theta$$

and denoting the matrix of $\wedge_k T$ by A_k we have from (51)

(53) $$A_k = (A_{\theta,\psi})$$

which is simply the $\binom{n}{k} \times \binom{n}{k}$-matrix of all $k \times k$-minors of the matrix A, arranged in an order that corresponds to the ordering of the basis elements $\{e_\psi\}$ of $\wedge_k(V)$. In general, lexicographic ordering of the e_ψ is a natural choice. To transform the Grassmann coordinates of $[W]$ we now simply observe that

(54) $$[[\wedge_k T][W]]_\psi = [\wedge_k T(W)]_\psi = [A_k] \cdot [W]_\psi$$

where $[A_k] = \{\alpha A_k : \alpha \neq 0\}$ and the last step of (54) is ordinary matrix multiplication. Hence $[A_k]$ is the matrix representation of $[\wedge_k T]$ and the transformation of subspaces induced by a linear map $T : V \to V$ is performed by the corresponding matrix multiplication of the Grassmann coordinates.

We can sum up the situation in the following diagrams:

$$\begin{array}{ccccc} & V & \xrightarrow{T} & V & \\ [\,]_B & \downarrow & & \downarrow & [\,]_B \\ & \mathbb{R}^n & \xrightarrow{A} & \mathbb{R}^n & \end{array} \tag{55}$$

$$\begin{array}{ccccc} & \wedge_k(V)/_\sim & \xrightarrow{[\wedge_k T]} & \wedge_k(V)/_\sim & \\ [\,]_\psi & \downarrow & & \downarrow & [\,]_\psi \\ & \mathbb{R}^{\binom{n}{k}}/_\sim & \xrightarrow{[A_k]} & \mathbb{R}^{\binom{n}{k}}/_\sim & \end{array} \tag{56}$$

By a choice of basis $\mathcal{B}$ for the space V the linear map $T : V \to V$ is represented (as usual) by the matrix operator $A : \mathbb{R}^n \to \mathbb{R}^n$ as depicted in (55). This induces a representation of the corresponding *linear subspace transformation operator* $[\wedge T] = \bigoplus_k [\wedge_k T]$ whose action on a k-dimensional subspace $[W]$ is represented by multiplication of its Grassmann coordinates $[W]_\psi$ by any one of the matrices $[A_k]$ according to (56).

Conclusion

We have demonstrated in a few pages the basic properties of exterior algebra and Grassmann coordinates, and sketched briefly how they can be applied to the study of vision. It seems to us that the advantage of doing so lies mainly in the possibility of achieving a unified and systematic description of a whole family of geometric interrelationships inherent in a 3-d scene. Each of these relations is often more efficiently described in its own *tailor-made* coordinate system, but the analysis of their interaction ought to benefit from a coherent, *universal* framework with no possible exceptions in its way of describing things.

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