Discrete Integration and the Fundamental Theorem

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Abstract

This paper introduces a computational framework - termed *discrete integration* - which presents an algebraic interpretation of integration that captures the combinatorial aspects of the fundamental theorem of calculus at a finite level. This facilitates discrete approximations of integrals that are evaluated over smooth geometric configurations, since it enables their numerical approximation within the same conceptual space. It also makes possible a discrete version of the differential and integral calculus over various function classes. We illustrate these ideas in the case of polynomial functions - establishing a scale-invariant way to compute with sums of polynomials defined on simplexes[1].

1 Discrete integration

We develop the computational framework of discrete integration in terms of what we call the *endomorphic integral*. At the time when we worked out these ideas we were not familiar with the work of Sobczyk in this field, but in retrospect we have come to realize that our ideas constitute a natural extension of his work into the arenas of complexes and endomorphisms, which explains the qualifier ‘endomorphic’ for the integral and differential that we introduce in Def. (5) and Def. (6).

Apart from their intrinsic mathematical interest, we feel that this kind of integral and differential are quite powerful as computational engineering tools, and we demonstrate their combinatorial power in Prop. (2), which we call *the combinatorial version of the fundamental theorem of calculus*.

2 Basic definitions

Consider a geometric algebra $G = G(E)$, and let $V = G^1(E)$ be the linear space of its 1-vectors. It is easy to verify the following

Prop. 1: The set of even permutations of n+1 objects acts naturally on the cartesian product V^{n+1}, and therefore induces an equivalence relation ~ on it.

Using this proposition, we can now make the following

Def. 1: The set S_n of n-simplices is defined as $S_n = V^{n+1}/\sim$.
The set S of all simplices is defined as $S = \bigcup_{n \geq 0} S_n$

1.Since we have not fixed a notation for the polynomial example, we cannot at this point supply any formulas on this part.

Def. 2: A *complex* is a formal finite linear combinations of simplices with integral coefficients. Thus the set C of all complexes is defined as

$$C = \bigoplus_{S} \mathbf{Z} = \bigoplus_{n \geq 0} C_n \quad , \tag{1}$$

where $\mathbf{Z}$ denotes the set of integers, and $C_n = \bigoplus_{S_n} \mathbf{Z}$.

3 The boundary operator

Def. 3: Consider an n-simplex $s \in S$, i.e. $s = [v_0, v_1, \ldots, v_n] \in S_n$, and let $s_i = [v_0, \ldots, \hat{v}_i, \ldots, v_n]$, where $\hat{v}_i$ indicates deletion of v_i .
The boundary operator ∂ is defined on S by:

$$\partial s = \sum_{i=0}^{n} (-1)^i s_i \tag{2}$$

and extended to C by linearity.

Note: It is a simple exercise to verify that $\partial \circ \partial = 0$

4 Connecting *C* with *G*

We will now define a geometric measure σ called the *directed content* of each simplex, a measure that has been introduced by Sobczyk. The directed content of a simplex captures both its direction and its magnitude in the form of a single multivector. We extend the measure σ to C by making the following

Def. 4: The $\mathbf{Z}$-linear mapping $\sigma : C \to G$ taking n-simplices to n-blades is defined by

$$\begin{aligned} \sigma[v_0] &= 1 \\ \sigma[v_0, v_1, \ldots, v_n] &= \frac{1}{n!}(v_1 - v_0) \wedge \ldots \wedge (v_n - v_0) \end{aligned} \tag{3}$$

Note: It is easily verified that σ is well defined, i.e. if π is an even permutation on $\{1, 2, \ldots, n\}$, then we have

$$\sigma[v_{\pi(0)}, v_{\pi(1)}, \ldots, v_{\pi(n)}] = \sigma[v_0, v_1, \ldots, v_n] \tag{4}$$

By a straight-forward calculation one also obtains $\sigma \circ \partial = 0$.

5 Definition of the endomorphic integral

With *End* (*G*) denoting the ring of endomorphisms of *G*, we start by introducing:

$$\Psi = Hom_{\mathbf{Z}}(C, G) \quad , \qquad \Omega = Hom_{\mathbf{Z}}(C, End(G)) \tag{5}$$

Def. 5: The *endomorphic integral* $\int$ is a mapping defined by the diagram

$$\begin{array}{ccccc} \Omega & \xrightarrow{\int} & \Psi & & \\ \Cup & & \Cup & & \\ \omega & \longmapsto & \int\omega : C & \longrightarrow & G \\ & & \Cup & & \Cup \\ & & c & \longmapsto & \int_c \omega = \sum_s c_s \int_s \omega = \sum_s c_s \omega(s)[\sigma(s)] \end{array}$$

Now, for $x \in G$, we let $\rho(x)$ denote multiplication with x from the right, i.e.

$$\begin{array}{ccccc} G & \xrightarrow{\rho} & End(G) & & \\ \Cup & & \Cup & & \\ x & \longmapsto & \rho(x) : G & \longrightarrow & G \\ & & \Cup & & \Cup \\ & & y & \longmapsto & yx \end{array}$$

Before we define the endomorphic differential, we introduce the sets

$$S^* = \{s \in S : \sigma(s) \text{ is invertible}\} \quad , \quad C^* = \bigoplus_{S^*} \mathbf{Z} \tag{6}$$

and the sets corresponding to (5):

$$\Psi^* = Hom_{\mathbf{Z}}(C^*, G) \quad , \quad \Omega^* = Hom_{\mathbf{Z}}(C^*, End(G)) \tag{7}$$

Def. 6: The *endomorphic differential* $d : \Omega \to \Omega^*$ is defined by

$$d\omega(s)[x] = \int_{\partial s} \omega \circ \rho(\sigma(s)^{-1} x) = \sum_i (-1)^i \omega(s_i)[\sigma(s_i)\sigma(s)^{-1} x] \tag{8}$$

We are now in a position to state the following combinatorial version of the fundamental theorem of calculus:

Prop. 2: The endomorphic integral and the endomorphic differential, are related to any complex and its boundary by the identity

$$\int_c d\omega = \int_{\partial c} \omega \tag{9}$$

Proof: From (8), which defines the endomorphic differential, we have directly:

$$\begin{aligned} \int_s d\omega &= (d\omega)(s)[\sigma(s)] = \sum_i (-1)^i \omega(s_i)[\sigma(s_i)] \\ &= \sum_i (-1)^i \int_{s_i} \omega = \int_{\partial s} \omega \end{aligned} \tag{10}$$

Since this holds for each participating simplex in the linear combination c, by linear extension it must hold for the complex c itself, which proves the theorem. <<>>