Combinatorial Aspects of Clifford Algebra

Lars Svensson and Ambjörn Naeve
Dept. of Mathematics / Dept. of Numerical Analysis and Computing Science
Royal Institute of Technology
100 44 Stockholm
Sweden
[larss@math.kth.se / amb@nada.kth.se]

Abstract

In this paper we focus on some combinatorial aspects of Clifford algebra and show how this algebra allows combinatorial theorems - like e.g. Sperner's lemma - to be "built into the algebraic background", and become part of the structure of the algebra itself. We also give an example of how cumbersome combinatorial proofs can be "mechanized" and carried out in a purely computational manner.

Introduction

In his monumental and groundbreaking *Ausdehnungslehre* [4] from 1844, Herman Grassmann set out to build an "algebra for everything" - an algebra which he illustrated by various geometric examples[1]. Being both far ahead of his time and on the outside of the academic mathematical community, Grassmann's ideas received little attention during his own lifetime. However, during the last years of Grassmann's life (late 1870s), his ideas were taken up by William Clifford [2], [3], who developed the algebra that today bears his name. In more recent times mathematicans and physicists - notably Marcel Riesz [8], Gian-Carlo Rota [1], [9], and David Hestenes [5], [6], [7] - have rediscovered and continued this development. Hestenes has focused on the geometric aspects of Clifford algebra - introducing the synonymous term *geometric algebra* - and shown how it provides a powerful geometric language that serves as a bridge between mathematics and physics.

In this paper we aim to connect with Grassmann's original ideas, and follow Rota [9] by focusing on the purely combinatorial aspects of Clifford algebra.

Some notation and background

Since we are only interested in combinatorial and algebraic aspects of Clifford algebra, we will allow our scalars to lie in an arbitrary commutative ring R with unit element. We will also take a slightly different point of view regarding the Clifford algebra and its interpretation.

Let X be a finite set which is totally ordered, i.e. $X = \{x_1, \ldots, x_n\}$, where $x_1 < x_2 < \ldots < x_n$. We will identify the k-base-blades $x_{1'} x_{2'} \ldots x_{k'}$ with the k-subsets $\{x_{1'}, \ldots, x_{k'}\}$ and denote the pseudoscalar $x_1 x_2 \ldots x_n$ by X. The ring-unit 1 is identified with the empty set $\varnothing$. We will view the Clifford algebra Cl(X) as the free R-module generated by the power-set $\wp(X)$ of all subsets of X, i.e. $\mathrm{Cl}(X) = \bigoplus_{\wp(X)} R$.

Note that if $X \to Y$ is a bijection, then Cl(X) is isomorphic to Cl(Y).

1. For instance, the today familiar concept of *vector* is an example of what Grassmann termed *evolution*. See [4], p. 46.

We will always assume that $x^2 = 1$, $\forall x \in X$. The set of k-vectors is denoted by $Cl^k(X)$. We observe that every bilinear map $Cl(X) \times Cl(X) \to Cl(X)$ is uniquely determined by its values on $\wp(X) \times \wp(X)$. Moreover, if P is a proposition, we will use (P) to denote 1 or 0 depending on whether P is true or false.

Let $A,B \in \wp(X)$. The following notation is used below:

Geometric product: $AB = \varepsilon A \Delta B$,
where $\varepsilon = \pm 1$ and Δ denotes symmetric difference.

Outer product: $A \wedge B = (A \cap B = \emptyset) AB$.

Left inner product: $A \angle B = (A \subseteq B) AB$.

Scalar product: $A * B = (A = B) AB$.

Reverse: $A^\dagger = (-1)^\varepsilon A$, where $\varepsilon = \binom{|A|}{2}$.

Complement: $\tilde{A} = AX^{-1}$.

All of these definitions are extended to Cl(X) by linearity.

Below we will need the following simple

Lemma 1: $x \angle (y \angle z) = (x \wedge y) \angle z$, $\forall x, y, z \in Cl(X)$.

Proof: By linearity, it is enough to verify this for base-blades, i.e. for $x = A$, $y = B$, $z = C$ where $A, B, C \in \wp(X)$. This is a simple exercise (using e.g. Venn diagrams), which we leave to the reader.

If Cl(X) and Cl(Y) are two Clifford algebras over the same ring, we call a linear and grade preserving map $T : Cl(X) \to Cl(Y)$ an *outermorphism* if $T(1) = 1$ and $T(x \wedge y) = Tx \wedge Ty$, $\forall x, y \in Cl(X)$. The obvious fact that an outermorphism on Cl(X) is uniquely determined by its values on X will be implicitely used below in all our definitions of various outermorphisms.

The *dual* $\tilde{T}$ of T is defined by $\tilde{T}(x) = T(xX)X^{-1}$.

We will also make use of the following fundamental theorem, due to David Hestenes[1]:

Prop. 1: If $T : Cl(X) \to Cl(Y)$ is an outermorphism with adjoint T^*, then
$x \angle Ty = T(T^* x \angle y)$.

1. Hestenes & Sobczyk [7], p.69, (1.14).

Clifford algebras and graphs

Let $G = (V, E)$ be a (undirected) graph with vertices $V = \{v_1, \ldots, v_n\}$ and edges $E = \{e_1, \ldots, e_m\}$. Choose a total ordering on V according to the given enumeration. This induces a direction on G, by letting an edge $e \in E$ that connects the vertices $v, v' \in V$ be directed from v to v' if $v < v'$.

We now form the two Clifford algebras Cl(V) and Cl(E) with $v_i^2 = e_j^2 = 1$, and for $v, v' \in V$, $e \in E$, we define the mapping $g : (V \cup E) \times V \to \{-1, 0, 1\}$ by

$g(v, v') = (v$ and v' are neighbors), and
$g(e, v) = (e$ ends at $v) - (e$ starts from $v)$.

Moreover, we define the following outermorphisms:

$$\begin{array}{ccl} Cl(V) & \xrightarrow{A} & Cl(V) \\ v & \longmapsto & \sum_{v' \in V} g(v, v')\, v' \text{, for } v \in V. \end{array}$$

$$\begin{array}{ccl} Cl(V) & \xrightarrow{\delta} & Cl(V) \\ v & \longmapsto & \mathrm{val}(v)\, v \text{, where } \mathrm{val}(v) \text{ is the valence of } v. \end{array}$$

$$\begin{array}{ccl} Cl(E) & \xrightarrow{\partial} & Cl(V) \\ e & \longmapsto & \sum_{v \in V} g(e, v)\, v \text{, for } e \in E. \end{array}$$

$$\begin{array}{ccl} Cl(V) & \xrightarrow{\partial^*} & Cl(E) \\ v & \longmapsto & \sum_{e \in E} g(e, v)\, e \text{, for } v \in V. \end{array}$$

Since $(\partial e) * v = \sum_{v' \in V} g(e, v') v' * v = g(e, v)$,
and $e * \partial^*(v) = e * \sum_{e' \in E} g(e', v) e' = g(e, v)$,

we see that ∂^* is the adjoint of ∂.

The *Laplacian* of the graph G is defined as the outermorphism

$\Delta = \partial \circ \partial^* : Cl(V) \to Cl(V)$.

We immediately obtain the following relation between Δ, δ and A:

Prop. 2: $\Delta = \delta - A$.

Proof: A direct computation gives
$(v) = \partial \circ \partial^*(v) = \partial \sum_{e \in E} g(e, v) e = \sum_{e \in E} \sum_{v' \in V} g(e, v) g(e, v') v' =$

$$= \sum_{e,v'} (g(e,v)g(e,v') = 1)\,v' \; - \; \sum_{e,v'} (g(e,v)g(e,v') = -1)\,v'$$
$$= \mathrm{val}(v)v - \sum_{v'} g(v,v')v' \; = \delta(v) - A(v)\,.$$

One also easily shows the following[1]

Prop. 3: $\mathrm{tr}\, A = 0$, $\mathrm{tr} \wedge^2 A = |E|$, $\mathrm{tr} \wedge^3 A = (-2)(\#\ 3\text{-cycles in } G)$.

Moreover, there is the following useful

Lemma 2: If the graph G = (V, E) contains a cycle, then $\partial E = 0$.

Proof: We identify E with $e_1 e_2 \ldots e_m$. Assume that G contains the cycle $v_1, v_2, \ldots, v_k, v_1$. But $v_k - v_1 = (v_k - v_{k-1}) + (v_{k-1} - v_{k-2}) + \ldots + (v_2 - v_1)$. Therefore $(v_2 - v_1) \wedge \ldots \wedge (v_k - v_{k-1}) \wedge (v_k - v_1) = 0$, which means that $\partial E = \partial(e_1 e_2 \ldots e_m) = \partial(e_1 \wedge e_2 \wedge \ldots \wedge e_m)$ contains a Λ-factor 0. Hence $\partial E = 0$.

The converse of lemma 2 is contained in

Lemma 3: Let G = (V, E) be a graph with $|V| = n, |E| = m$ and write $V = v_1 v_2 \ldots v_n$. If F is a (n-1)-subset of E such that the edges of F form a tree (i.e. no cycles), then we have $v \wedge (\partial F) = \pm V$, $\forall v \in V$, where the sign is independent of v.

Proof: To show that $v \wedge (\partial F)$ is independent of v, it is enough to show that $v' \wedge (\partial F) = v'' \wedge (\partial F)$ if v' and v'' are connected by an edge - say f - in F. But if $\partial f = v'' - v'$, it follows that $(v'' - v') \wedge (\partial F) = (\partial f) \wedge (\partial F) = \partial (f \wedge F) = 0$. Moreover, since F is a tree and $|F| = n - 1$, F is a spanning tree for G and hence contains all the vertices of V. Let v have neighbors $v', v'', \ldots$. Then we have $v \wedge (v' - v) \wedge (v'' - v) \wedge \ldots = v \wedge v' \wedge v'' \wedge \ldots$ and hence, by applying the same argument to $v', v'', \ldots$ we get $v \wedge (\partial F) = \pm V$, which finishes the proof.

We note in general that if $F \subseteq E$, then two vertices v' and v'' are in the same component of F if and only if $(v'' - v') \wedge (\partial F) = 0$.

We are now ready to prove a well-known and nice result about the number of spanning trees in a given graph. Our proof shows the computational power inherent in applying Clifford algebra to graph theory.

Prop. 4: If $\Delta = \partial \circ \partial^* : Cl(V) \to Cl(V)$ is the Laplacian outermorphism, then $\Delta^{adj} = N\mathfrak{I}$, where N is the number of spanning trees in G, and $\mathfrak{I} : Cl(V) \to Cl(V)$ is given by $\mathfrak{I}(v) = \sum_{v' \in V} v' = s$, $\forall v \in V$.

1. the proof of which is left to the reader.

Proof: From the definition of the adjoint, we have $\Delta^{adj} = \tilde{\Delta^*} = \tilde{\Delta}$. Moreover, $\Delta^{adj}(y) = \sum_{x \in V}(x*(\Delta^{adj}y))x$, $\forall y \in V$. Hence we must show that $x*(\Delta^{adj}y) = N$, $\forall x, y \in V$.

Applying the definitions, we get

$x*(\Delta^{adj}y) = x*(\tilde{\Delta}y) = x*(\Delta(yV)V^\dagger) = \langle x(\Delta(yV)V^\dagger)\rangle_0$. Now, since Δ is grade-preserving, $x\Delta(yV)$ is a pseudoscalar and thus commutes with $V^\dagger$. Hence we have

$$\begin{aligned} x*(\Delta^{adj}y) &= \langle x\Delta(yV)V^\dagger\rangle_0 = \langle V^\dagger x\Delta(yV)\rangle_0 = (V^\dagger x)*(\Delta(yV)) \\ &= (V^\dagger x)*\partial(\partial^*(yV)) = \partial^*(V^\dagger x)*\partial^*(yV) \quad . \end{aligned}$$

Now, $V^\dagger x$ and yV are $(|V|-1)$-vectors in Cl(V), and since ∂ and ∂^* are grade-preserving, $\partial^*(V^\dagger x)$ and $\partial^*(yV)$ are $(|V|-1)$-vectors in Cl(E). Expanding these vectors, and summing over all $(|V|-1)$-vectors $F \subseteq E$, we get

$$\begin{aligned} x*(\Delta^{adj}y) &= \partial^*(V^\dagger x)*\partial^*(yV) = (\sum_F(\partial^*(V^\dagger x)*F)F^\dagger)*(\sum_F(\partial^*(yV)*F^\dagger)F) \\ &= \sum_F((V^\dagger x)*\partial F)((yV)*\partial F^\dagger) = \sum_F((xV)*\partial F^\dagger)((yV)*\partial F^\dagger) \\ &= \sum_F((xV)*\partial F)((yV)*\partial F) = \sum_F(\partial F\angle(x\angle V))(\partial F\angle(y\angle V)) \quad . \end{aligned}$$

Applying lemma 1, we get

$$x*(\Delta^{adj}y) = \sum_F(\partial F\wedge x)V(\partial F\wedge y)V = V^2\sum_F(x\wedge\partial F)(y\wedge\partial F) \quad .$$

Now, by lemma 2, $\partial F = 0$ unless F is a spanning tree, and, by lemma 3, in this case we have $x\wedge\partial F = y\wedge\partial F = \pm V$. Hence, remembering that N denotes the number of spanning trees in G, we have $x*(\Delta^{adj}y) = V^2\sum_{F \in SpT}(x\wedge\partial F)^2 = V^4N = N$, which finishes the proof.

Combinatorial algebraic topology

Let V denote an arbitrary finite set, and let $A = \{a_1, \ldots, a_n\}$. Let **Z** be the ring of scalars and consider the two Clifford algebras Cl(V) and Cl(A) over **Z** with $v^2 = 1, v \in V$. and $a_i^2 = 1, a_i \in A$.

A *labeling* of the elements of V by the elements of A is a mapping $\lambda: V \to A$.
Any such labeling is naturally extended to an outermorphism λ: Cl(V) $\to$ Cl(A).

Define the outermorphisms ∂_V : Cl(V) $\to$ Cl(V) and ∂_A : Cl(A) $\to$ Cl(A) by $\partial_V(x) = (\sum_{v \in V} v)\angle x$ and $\partial_A(y) = (\sum_{a \in A} a)\angle y$ respectively.

We now define the *content* mapping Cont and the *index* mapping Ind by

$$\begin{array}{ccc} Cl^n(V) & \xrightarrow{\text{Cont}} & \mathbf{Z} \\ x & \longmapsto & A^\dagger * \lambda(x) \end{array}$$

$$\begin{array}{ccc} Cl^n(V) & \xrightarrow{\text{Ind}} & \mathbf{Z} \\ x & \longmapsto & \alpha\circ\lambda\circ\partial_V(x) \end{array}$$

where

$$\begin{array}{ccc} Cl(A) & \xrightarrow{\alpha} & Cl(A) \\ y & \longmapsto & (a_{n-1}\ldots a_1)\angle y\ . \end{array}$$

Note that $\alpha(Cl^{n-1}(A)) \subseteq Z$. We now have the following

Prop. 5: $\mathrm{Ind} = \alpha\circ\lambda\circ\partial_V = \alpha\circ\partial_A\circ\lambda = (-1)^{n-1}\mathrm{Cont}$.

Proof: Consider the n-blade $x = v_1v_2\ldots v_n$ in Cl(V) and let $\hat{v}$ denote deletion of v. We now have

$$\begin{aligned}\alpha\circ\lambda\circ\partial_V(x) &= \alpha\circ\lambda((v_1+\ldots+v_n)\angle(v_1\ldots v_n)) = \alpha\circ\lambda(\sum_j (-1)^{j-1}v_1\ldots\hat{v}_j\ldots v_n)\\ &= \alpha(\mathbf{\Sigma}(-1)^{j-1}\lambda(v_1)\wedge\ldots\wedge\lambda(\hat{v}_j)\wedge\ldots\wedge\lambda(v_n))\\ &= \mathbf{\Sigma}(-1)^{j-1}(a_{n-1}\ldots a_1)\angle(\lambda(v_1)\wedge\ldots\wedge\lambda(\hat{v}_j)\wedge\ldots\wedge\lambda(v_n))\ .\end{aligned}$$

Moreover, we have

$$\begin{aligned}\alpha\circ\partial_A\circ\lambda(x) &= (a_{n-1}\ldots a_1)\angle((a_1+\ldots+a_n)\angle(\lambda(v_1)\wedge\ldots\wedge\lambda(v_n)))\\ &= ((a_{n-1}\ldots a_1)\wedge(a_1+\ldots+a_n))\angle(\lambda(v_1)\wedge\ldots\wedge\lambda(v_n))\\ &= (a_{n-1}\ldots a_1a_n) * \lambda(x) = (-1)^{n-1}\mathrm{Cont}(x)\ .\end{aligned}$$

We must consider the following two different possibilities:

1) If λ is injective on $v_1, \ldots, v_n$, we can WLOG[1] assume that $\lambda(v_i) = a_i$, $\forall i$. We then have

$$\begin{aligned}\alpha\circ\lambda\circ\partial_V(x) &= \mathbf{\Sigma}(-1)^{j-1}(a_{n-1}\ldots a_1)\angle(\lambda(v_1)\wedge\ldots\wedge\lambda(\hat{v}_j)\wedge\ldots\wedge\lambda(v_n))\\ &= (-1)^{n-1}(a_{n-1}\ldots a_1)\angle(a_1\wedge\ldots\wedge a_{n-1}) = (-1)^{n-1}\ ,\end{aligned}$$

$$\mathrm{Cont}(x) = (-1)^{n-1}(a_na_{n-1}\ldots a_1) * (a_1\ldots a_n) = (-1)^{n-1}\ .$$

2) If λ is not injective on $v_1, \ldots, v_n$, we can WLOG assume that $\lambda(v_1) = \lambda(v_2)$. In this case we get

1. Without Loss Of Generality.

$$
\begin{aligned}
\alpha\circ\lambda\circ\partial_V(x) &= \boldsymbol{\Sigma}(-1)^{j-1}(a_{n-1}\dots a_1)\angle(\lambda(v_1)\wedge\dots\wedge\lambda(\hat{v}_j)\wedge\dots\wedge\lambda(v_n)) \\
&= (a_{n-1}\dots a_1)\angle(\lambda(v_2)\wedge\lambda(v_3)\wedge\dots\wedge\lambda(v_n)) \\
&\quad -(a_{n-1}\dots a_1)\angle(\lambda(v_1)\wedge\lambda(v_3)\wedge\dots\wedge\lambda(v_n)) \\
&= 0\ ,
\end{aligned}
$$

$$
\mathrm{Cont}(x) = (a_n\dots a_1) * (\lambda(v_1)\wedge\lambda(v_2)\wedge\dots\wedge\lambda(v_n)) = 0\,.
$$

This proves the proposition.

Hence we have the following commutative diagram:

$$
\begin{tikzcd}
 & \mathrm{Cl}^n(V) \arrow[r, "\partial_V"] \arrow[dl, "\mathrm{Cont}"'] \arrow[d, "\lambda"'] \arrow[drr, "\mathrm{Ind}"] & \mathrm{Cl}^{n-1}(V) \arrow[r, "\lambda"] & \mathrm{Cl}^{n-1}(A) \arrow[d, "\alpha"] \\
\mathbf{Z} \arrow[rrr, bend right, "(-1)^{n-1}id"'] & \mathrm{Cl}^n(A) \arrow[r, "\partial_A"] & \mathrm{Cl}^{n-1}(A) \arrow[r, "\alpha"] & \mathbf{Z}
\end{tikzcd}
$$

Remark: From this index theorem there follows immediately several basic results in combinatorial topology, notably the so called Sperner's lemma.

Koszul complexes

We like to finish by presenting an example from algebra where we hope to illustrate that it may sometimes be a good idea not to restrict the scalars of the Clifford algebra to only **R** or **C**, but to allow them to lie in arbitrary commutative rings with unit.

Let I and J be two ideals in such a ring R generated by $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_m\}$ respectively. We also assume that $J \subseteq I$, i.e. that $y_j = \boldsymbol{\Sigma} c_{ji}x_i$ for some c_{ji} in R. Below we are going to use Clifford algebra to define the famous *Koszul complexes* $K(x_1, \dots, x_n)$ and $K(y_1, \dots, y_m)$, which have a lot of applications in algebra and differential topology.

We will also give an almost trivial proof of the well-known fact that there exists a chain-complex morphism $K(y_1, \dots, y_m) \to K(x_1, \dots, x_n)$, which is an isomorphism of chain-complexes if $I = J$.

We start by introducing sets $E = \{e_1, \dots, e_n\}$ and $F = \{f_1, \dots, f_m\}$ and their associated Clifford Algebras Cl(E) and Cl(F) over the ring R. We may assume that $e_i^2 = f_j^2 = 1,\ \forall i, j$.

Let $T : \mathrm{Cl}(F) \to \mathrm{Cl}(E)$ be the outermorphism defined by $Tf_j = \boldsymbol{\Sigma} c_{ji}e_i$, and denote by $T^* : \mathrm{Cl}(E) \to \mathrm{Cl}(F)$ the outermorphism defined by $T^*e_i = \boldsymbol{\Sigma} c_{ji}f_j$.

Then we have $T^*e_i * f_j = c_{ji} = e_i * Tf_j$, so T^* is the adjoint of T. Moreover, if $e = x_1e_1 + \dots x_ne_n$ and $f = y_1f_1 + \dots y_mf_m$, we obtain $T^*e = f$.

Let us introduce the two “boundary” mappings $d_e : Cl(E) \to Cl(E)$ and $d_f : Cl(F) \to Cl(F)$ by $d_e(x) = e \angle x$, $\forall x \in Cl(E)$ respectively $d_f(y) = f \angle y$, $\forall y \in Cl(F)$. Then we have $d_e \circ d_e(x) = e \angle (e \angle x) = (e \wedge e) \angle x = 0$ and similarly for $d_f \circ d_f(y)$, which means that $d_e^2 = d_f^2 = 0$.

Hence the pairs $(Cl(E), d_e)$ and $(Cl(F), d_f)$ become two chain-complexes, which by definition are the Koszul complexes - usually denoted $K(x_1, \ldots, x_n)$ respectively $K(y_1, \ldots, y_m)$.

Finally, we prove that T is a morphism of chain-complexes. By definition this means that the diagram below commutes.

$$\begin{array}{ccc} Cl(F) & \xrightarrow{d_f} & Cl(F) \\ T\downarrow & & \downarrow T \\ Cl(E) & \xrightarrow{d_e} & Cl(E) \end{array}$$

But if $y \in Cl(F)$, we have by Hestenes’ theorem (proposition 1): $(d_e \circ T) y = e \angle T y = T(T^* e \angle y) = T(f \angle y) = (T \circ d_f) y$, which shows the commutativity. Moreover, if $\det(c_{ji})$ is invertible in R, then T is an isomorphism of the Koszul complexes.

Conclusions and future work

In this paper we have given some hints as to the many possibilities that are inherent in using generalized versions of Clifford algebra outside of the conventional domains - such as e.g. geometry - where this algebra is usually applied. We hope that our choice of examples, although few and simple - have convinced the reader that it is a fruitful idea to use the computational power of Clifford algebra also in fields like combinatorics and algebra. In future papers we hope to give more substantial results that will prove the strength of this approach.

References

[1] Barabei, M. & Brini, A. & Rota, G.-C., *On the Exterior Calculus of Invariant Theory*, Journal of Algebra, Vol. 96, pp.120-160, 1985.

[2] Clifford, W. K., *Applications of Grassmann’s extensive algebra*, American Journal of Mathematics, Vol. I, pp. 350-358, 1878.

[3] Clifford, W. K., *Mathematical Papers*, (ed. Tucker), Macmillan, London, 1882.

[4] Grassmann, H., *Die Lineale Ausdehnungslehre – ein neuer Zweig der Mathematik*, Leipzig, 1844. (Translated to English by Lloyd C. Kannenberg, *A New Branch of Mathematics*, Open Court, 1995).

[5] Hestenes, D., *Space-Time Algebra*, Gordon & Breach, New York, 1966.

[6] Hestenes, D., *New Foundations For Classical Mechanics*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1993 (1986).

[7] Hestenes, D. & Sobczyk, G., *Clifford Algebra to Geometric Calculus: A Unified Language for Mathematics and Physics*, D. Reidel Publishing Company, Dordrecht, The Netherlands, 1992 (1984).

[8] Riesz, M., *Clifford Numbers and Spinors*, Lecture Series No. 38, The Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, 1958.

[9] Rota, G.-C. & Stein, J., *Applications of Cayley algebras*, Teorie Combinatoire, Tomo II, p. 71, Accademia Nazionale Dei Lincei, Roma, 1976.